

# Stable determination of coefficients in the dynamical anisotropic Schrödinger equation from the Dirichlet-to-Neumann map

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## Abstract

In this paper we are interested in establishing stability estimates in the inverse problem of determining on a compact Riemannian manifold the electric potential or the conformal factor in a Schrödinger equation with Dirichlet data from measured Neumann boundary observations. This information is enclosed in the dynamical Dirichlet-to-Neumann map associated to the Schrödinger equation. We prove in dimension  $n \geq 2$  that the knowledge of the Dirichlet-to-Neumann map for the Schrödinger equation uniquely determines the electric potential and we establish Hölder-type stability estimates in determining the potential. We prove similar results for the determination of a conformal factor close to 1.

**Keywords:** Stability estimates, Schrödinger inverse problem, Dirichlet-to-Neumann map.

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## 1 Introduction and main results

This paper is devoted to the study of the following inverse boundary value problem: given a Riemannian manifold with boundary determine the potential or the conformal factor of the metric in a dynamical Schrödinger equation from the observations made at the boundary. Let  $(\mathcal{M}, g)$  be a compact Riemannian manifold with boundary  $\partial\mathcal{M}$ . All manifolds will be assumed smooth (which means  $\mathcal{C}^\infty$ ) and oriented. We denote by  $\Delta_g$  the Laplace-Beltrami operator associated to the metric  $g$ . In local coordinates,  $g(x) = (g_{jk})$ ,  $\Delta_g$  is given by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left( \sqrt{\det g} g^{jk} \frac{\partial}{\partial x_k} \right). \quad (1.1)$$

Here  $(g^{jk})$  is the inverse of the metric  $g$  and  $\det g = \det(g_{jk})$ . Let us consider the following initial boundary value problem for the Schrödinger equation with bounded electric potential  $q \in L^\infty(\mathcal{M})$

$$\begin{cases} (i\partial_t + \Delta_g + q(x)) u = 0, & \text{in } (0, T) \times \mathcal{M} \\ u(0, \cdot) = 0, & \text{in } \mathcal{M} \\ u = f, & \text{on } (0, T) \times \partial\mathcal{M} \end{cases} \quad (1.2)$$

where  $f \in H^1((0, T) \times \partial\mathcal{M})$ . Denote by  $\nu$  the outward normal vector field along the boundary  $\partial\mathcal{M}$ , so that  $\sum_{j,k=1}^n g^{jk} \nu_j \nu_k = 1$ . Further, we may define the dynamical Dirichlet-to-Neumann map  $\Lambda_{g,q}$  associated to the Schrödinger equation by

$$\Lambda_{g,q} f = \sum_{j,k=1}^n \nu_j g^{jk} \frac{\partial u}{\partial x_k} \Big|_{(0,T) \times \partial\mathcal{M}}. \quad (1.3)$$

Unique determination of the metric  $g = (g_{jk})$  from the knowledge of the Dirichlet-to-Neumann map  $\Lambda_{g,q}$  is hopeless: as was noted in [33] in the case of the wave equation, the Dirichlet-to-Neumann map is invariant under a gauge transformation of the metric  $g$ . Namely, if one pulls back the metric  $g$  by a diffeomorphism  $\Psi : \mathcal{M} \rightarrow \mathcal{M}$  which is the identity on the boundary  $\Psi|_{\partial\mathcal{M}} = \text{Id}$  into a new metric  $\Psi^*g$ , one has  $\Lambda_{\Psi^*g,q} = \Lambda_{g,q}$ . The inverse problem has therefore to be formulated modulo the gauge invariance. However, we will restrict our inverse problem to a conformal class of metrics (for which there is no gauge invariance): knowing  $\Lambda_{cg,q}$ , can one determine the conformal factor  $c$  and the potential  $q$ ?

In the case of the Schrödinger equation, Avdonin and Belishev gave an affirmative answer to this question for smooth metrics conformal to the Euclidean metric in [3]. Their approach is based on the boundary control method introduced by Belishev [5] and uses in an essential way a unique continuation property. Because of the use of this qualitative property, it seems unlikely that the boundary control method would provide accurate stability estimates. More precisely, when  $\mathcal{M}$  is a bounded domain of  $\mathbb{R}^n$ , and  $\varrho, q \in C^2(\overline{\mathcal{M}})$  are real functions, Avdonin and Belishev [3] show that for any fixed  $T > 0$  the response operator (or the Neumann-to-Dirichlet map) of the Schrödinger equation  $(i\varrho\partial_t u + \Delta u - qu) = 0$  uniquely determines the coefficients  $\varrho$  and  $q$ . The problem is reduced to recovering  $\varrho, q$  from the boundary spectral data. The spectral data are extracted from the response operator by the use of a variational principle.

The uniqueness in the determination of a time-dependent electromagnetic potential, appearing in a Schrödinger equation in a domain with obstacles, from the Dirichlet-to-Neumann map was proved by Eskin [17]. The main ingredient in his proof is the construction of geometrical optics solutions. In [2], Avdonin, Lenhart and Protopopescu use the so-called BC (boundary control) method to prove that the Dirichlet-to-Neumann map determines the time-independent electrical potential in a one dimensional Schrödinger equation.

The analogue problem for the wave equation has a long history. Unique determination of the metric goes back to Belishev and Kurylev [6] using the boundary

control method and involves works of Katchlov, Kurylev and Lassas [25], Kurylev and Lassas [27]) and Anderson, Katchalov, Kurylev, Lassas and Taylor [1]. In fact, Katchalov, Kurylev, Lassas and Mandache proved that the determination of the metric from the Dirichlet-to-Neumann map was equivalent for the wave and Schrödinger equations (as well as other related inverse problems) in [26]. Identifiability of the potential was proved by Rakesh and Symes [30] in the Euclidian case ( $g = e$ ) using complex geometrical optics solutions concentrating near lines with any direction  $\omega \in \mathbb{S}^{n-1}$  to prove that  $\Lambda_{e,q}$  determines  $q(x)$  uniquely in the wave equation. This result was generalized by Ramm and Sjöstrand [31] and Eskin [18, 19] to the case of  $q$  depending on space and time. Isakov [22] also considered the simultaneous determination of a potential and a damping coefficient.

As for the stability of the wave equation in the Euclidian case, we also refer to [35] and [23]; in those papers, the Dirichlet-to-Neumann map was considered on the whole boundary. Isakov and Sun [23] proved that the difference in some subdomain of two coefficients is estimated by an operator norm of the difference of the corresponding local Dirichlet-to-Neumann maps, and that the estimate is of Hölder type. Bellassoued, Jellali and Yamamoto [10] considered the inverse problem of recovering a time independent potential in the hyperbolic equation from the partial Dirichlet-to-Neumann map. They proved a logarithm stability estimate. Moreover in [29] it is proved that if an unknown coefficient belongs to a given finite dimensional vector space, then the uniqueness follows by a finite number of measurements on the whole boundary. In [7], Bellassoued and Benjoud used complex geometrical optics solutions concentrating near lines in any direction to prove that the Dirichlet-to-Neumann map determines uniquely the magnetic field induced by a magnetic potential in a magnetic wave equation.

In the case of the anisotropic wave equation, the problem of establishing stability estimates in determining the metric was studied by Stefanov and Uhlmann in [33, 34] for metrics close to Euclidean and generic simple metrics. In a previous paper [11], the authors also proved stability estimates for the wave equation in determining a conformal factor close to 1 and time independent potentials in simple geometries. We refer to this paper for a longer bibliography in the case of the wave equation.

The inverse problem for the (dynamical) Schrödinger equation seems to have been a little bit less studied. In the Euclidean case, there are extensive results by Bellassoued and Choulli [8] where a Lipschitz stability estimate was proven for time independent magnetic potentials. The stability problem in determining a time independent potential in a Schrödinger equation from a single boundary measurement was studied by Baudouin and Puel [4]. They established Lipschitz stability estimates by a method based essentially on an appropriate Carleman inequality. In

the above mentioned papers, the main assumption is that the part of the boundary where the measurement is made must satisfy a geometrical condition (related to geometric optics condition insuring observability). Recently, Bellassoued and Choulli showed in [9] that this geometric condition can be relaxed provided that the potential is known near the boundary. The key idea was the following : the authors used an FBI transform to change the Schrödinger equation near the boundary into a heat equation for which one can use a useful Carleman inequality involving a boundary term and without any geometric condition.

The main goal of this paper is to study the stability of the inverse problem for the dynamical anisotropic Schrödinger equation. We follow the same strategy as in [11] inspired by the works of Dos Santos Ferreira, Kenig, Salo and Uhlmann [16], Stefanov and Uhlmann [33, 34] and Bellassoued and Choulli [8].

### 1.1 Weak solutions of the Schrödinger equation

First, we will consider the initial-boundary value problem for the Schrödinger equation on a manifold with boundary (1.2). This initial boundary value problem corresponds to an elliptic operator  $-\Delta_g$  given by (1.1). We will develop an invariant approach to prove existence and uniqueness of solutions and to study their regularity properties.

Before stating our first main result, we recall the following preliminaries. We refer to [24] for the differential calculus of tensor fields on a Riemannian manifold. Let  $(\mathcal{M}, g)$  be an  $n$ -dimensional,  $n \geq 2$ , compact Riemannian manifold, with smooth boundary and smooth metric  $g$ . Fix a coordinate system  $x = [x_1, \dots, x_n]$  and let  $\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]$  be the corresponding tangent vector fields. For  $x \in \mathcal{M}$ , the inner product and the norm on the tangent space  $T_x \mathcal{M}$  are given by

$$g(X, Y) = \langle X, Y \rangle_g = \sum_{j,k=1}^n g_{jk} \alpha_j \beta_k,$$

$$|X|_g = \langle X, X \rangle_g^{1/2}, \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}.$$

If  $f$  is a  $\mathcal{C}^1$  function on  $\mathcal{M}$ , the gradient of  $f$  is the vector field  $\nabla_g f$  such that

$$X(f) = \langle \nabla_g f, X \rangle_g$$

for all vector fields  $X$  on  $\mathcal{M}$ . This reads in coordinates

$$\nabla_g f = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \quad (1.4)$$

The metric tensor  $g$  induces the Riemannian volume  $dv_g^n = (\det g)^{1/2} dx_1 \wedge \cdots \wedge dx_n$ . We denote by  $L^2(\mathcal{M})$  the completion of  $\mathcal{C}^\infty(\mathcal{M})$  endowed with the usual inner product

$$\langle f_1, f_2 \rangle = \int_{\mathcal{M}} f_1(x) \overline{f_2(x)} dv_g^n, \quad f_1, f_2 \in \mathcal{C}^\infty(\mathcal{M}).$$

The Sobolev space  $H^1(\mathcal{M})$  is the completion of  $\mathcal{C}^\infty(\mathcal{M})$  with respect to the norm  $\|\cdot\|_{H^1(\mathcal{M})}$ ,

$$\|f\|_{H^1(\mathcal{M})}^2 = \|f\|_{L^2(\mathcal{M})}^2 + \|\nabla_g f\|_{L^2(\mathcal{M})}^2.$$

The normal derivative is

$$\partial_\nu u := \nabla_g u \cdot \nu = \sum_{j,k=1}^n g^{jk} \nu_j \frac{\partial u}{\partial x_k} \quad (1.5)$$

where  $\nu$  is the unit outward vector field to  $\partial\mathcal{M}$ . Moreover, using covariant derivatives (see [20]), it is possible to define coordinate invariant norms in  $H^k(\mathcal{M})$ ,  $k \geq 0$ .

Before stating our main results on the inverse problem, our interest will focus on the study of the initial boundary problem (1.2), when  $u$  is a weak solution in the class  $\mathcal{C}(0, T; H^1(\mathcal{M})) \cap \mathcal{C}^1(0, T; H^{-1}(\mathcal{M}))$ . The following theorem gives conditions on  $f$  and  $q$ , which guarantee uniqueness and continuous dependence on the data of the solutions of the Schrödinger equation (1.2) with non-homogenous Dirichlet boundary condition.

**Theorem 1** *Let  $T > 0$  be given. Suppose that  $f \in H^1((0, T) \times \partial\mathcal{M})$  and  $q \in W^{1,\infty}(\mathcal{M})$ . Then the unique solution  $u$  of (1.2) satisfies*

$$u \in \mathcal{C}(0, T; H^1(\mathcal{M})) \cap \mathcal{C}^1(0, T; H^{-1}(\mathcal{M})), \quad (1.6)$$

$$\partial_\nu u \in L^2((0, T) \times \partial\mathcal{M}). \quad (1.7)$$

Furthermore, there is a constant  $C = C(T, \mathcal{M}) > 0$  such that

$$\|\partial_\nu u\|_{L^2((0,T) \times \partial\mathcal{M})} \leq C \|f\|_{H^1((0,T) \times \partial\mathcal{M})}. \quad (1.8)$$

The Dirichlet-to-Neumann map  $\Lambda_{g,q}$  defined by (1.3) is therefore continuous and we denote by  $\|\Lambda_{g,q}\|$  its norm in  $\mathcal{L}(H^1((0, T) \times \partial\mathcal{M}), L^2((0, T) \times \partial\mathcal{M}))$ .

Theorem 1 gives a rather comprehensive treatment of the regularity problem for (1.2) with stronger boundary condition  $f$ . Moreover, our treatment clearly shows that a regularity for  $f \in H^1((0, T) \times \partial\mathcal{M})$  is sufficient to obtain the desired interior regularity of  $u$  on  $(0, T) \times \mathcal{M}$  while the full strength of the assumption  $f \in H^1((0, T) \times \partial\mathcal{M})$  is used to obtain the desired boundary regularity for  $\partial_\nu u$  and then the continuity of the Dirichlet-to-Neumann map  $\Lambda_{g,q}$ .

## 1.2 Stable determination

In this section we state the main stability results. Let us first introduce the admissible class of manifolds for which we can prove uniqueness and stability results in our inverse problem. For this we need the notion of simple manifolds [34].

Let  $(\mathcal{M}, g)$  be a Riemannian manifold with boundary  $\partial\mathcal{M}$ , we denote by  $D$  the Levi-Civita connection on  $(\mathcal{M}, g)$ . For a point  $x \in \partial\mathcal{M}$ , the second quadratic form of the boundary

$$\Pi(\theta, \theta) = \langle D_\theta \nu, \theta \rangle_g, \quad \theta \in T_x(\partial\mathcal{M})$$

is defined on the space  $T_x(\partial\mathcal{M})$ . We say that the boundary is strictly convex if the form is positive-definite for all  $x \in \partial\mathcal{M}$ .

**Definition 1** *We say that the Riemannian manifold  $(\mathcal{M}, g)$  (or that the metric  $g$ ) is simple in  $\mathcal{M}$ , if  $\partial\mathcal{M}$  is strictly convex with respect to  $g$ , and for any  $x \in \mathcal{M}$ , the exponential map  $\exp_x : \exp_x^{-1}(\mathcal{M}) \rightarrow \mathcal{M}$  is a diffeomorphism. The latter means that every two points  $x, y \in \mathcal{M}$  are joined by a unique geodesic smoothly depending on  $x$  and  $y$ .*

Note that if  $(\mathcal{M}, g)$  is simple, one can extend it to a simple manifold  $\mathcal{M}_1$  such that  $\mathcal{M}_1 \supset \overline{\mathcal{M}}$ .

Let us now introduce the admissible set of potentials  $q$  and the admissible set of conformal factors  $c$ . Let  $M_0 > 0$ ,  $k \geq 1$  and  $\varepsilon > 0$  be given, set

$$\mathcal{Q}(M_0) = \left\{ q \in W^{1,\infty}(\mathcal{M}), \|q\|_{W^{1,\infty}(\mathcal{M})} \leq M_0 \right\}, \quad (1.9)$$

and

$$\begin{aligned} \mathcal{C}(M_0, k, \varepsilon) = \\ \left\{ c \in \mathcal{C}^\infty(\mathcal{M}), c > 0 \text{ in } \overline{\mathcal{M}}, \|1 - c\|_{\mathcal{C}^1(\mathcal{M})} \leq \varepsilon, \|c\|_{\mathcal{C}^k(\mathcal{M})} \leq M_0 \right\}. \end{aligned} \quad (1.10)$$

The main results of this paper are as follows.

**Theorem 2** *Let  $(\mathcal{M}, g)$  be a simple compact Riemannian manifold with boundary of dimension  $n \geq 2$  and let  $T > 0$ . There exist constants  $C > 0$  and  $s \in (0, 1)$  such that for any  $q_1, q_2 \in \mathcal{Q}(M_0)$ ,  $q_1 = q_2$  on  $\partial\mathcal{M}$ , we have*

$$\|q_1 - q_2\|_{L^2(\mathcal{M})} \leq C \|\Lambda_{g,q_1} - \Lambda_{g,q_2}\|^s \quad (1.11)$$

where  $C$  depends on  $\mathcal{M}, T, M_0, n, \alpha$  and  $s$ .

By Theorem 2, we can readily derive the following uniqueness result

**Corollary 1** *Assume that  $T > 0$ . Let  $q_1, q_2 \in \mathcal{Q}(M_0)$ ,  $q_1 = q_2$  on  $\partial\mathcal{M}$ . Then  $\Lambda_{g,q_1} = \Lambda_{g,q_2}$  implies  $q_1 = q_2$  everywhere in  $\mathcal{M}$ .*

**Theorem 3** *Let  $(\mathcal{M}, g)$  be a simple compact Riemannian manifold with boundary of dimension  $n \geq 2$  and let  $T > 0$ . There exist  $k \geq 1$ ,  $\varepsilon > 0$ ,  $0 < s < 1$  and  $C > 0$  such that for any  $c \in \mathcal{C}(M_0, k, \varepsilon)$  with  $c = 1$  near the boundary  $\partial\mathcal{M}$ , the following estimate holds true*

$$\|1 - c\|_{L^2(\mathcal{M})} \leq C \|\Lambda_g - \Lambda_{cg}\|^s \quad (1.12)$$

where  $C$  depends on  $\mathcal{M}$ ,  $M_0$ ,  $n$ ,  $\varepsilon$ ,  $k$  and  $s$ .

By Theorem 3, we can readily derive the following uniqueness result

**Corollary 2** *Let  $(\mathcal{M}, g)$  be a simple compact Riemannian manifold with boundary of dimension  $n \geq 2$  and let  $T > 0$ . There exist  $k \geq 1$ ,  $\varepsilon > 0$ , such that for any  $c \in \mathcal{C}(M_0, k, \varepsilon)$  with  $c = 1$  near the boundary  $\partial\mathcal{M}$ , we have that  $\Lambda_{cg} = \Lambda_g$  implies  $c = 1$  everywhere in  $\mathcal{M}$ .*

Our proof is inspired by techniques used by Stefanov and Uhlmann [34], and Dos Santos Ferreira-Kenig-Salo-Uhlmann [16] which prove uniqueness theorems for an inverse problem related to an elliptic equation. Their idea in turn goes back to the pioneering work of Calderón [13]. We also refer to Bukhgeim and Uhlmann [12], Cheng and Yamamoto [15], Hech-Wang [21] and Uhlmann [36] as a survey.

The outline of the paper is as follows. In section 2 we collect some of the formulas needed in the paper. In section 3 we study the Cauchy problem for the Schrödinger equation and we prove Theorem 1. In section 4 we construct special geometrical optic solutions to Schrödinger equations. In section 5 and 6, we establish stability estimates for related integrals over geodesics crossing  $\mathcal{M}$  and prove our main results.

## 2 Geodesical ray transform on a simple manifold

In this section we first collect some formulas needed in the rest of this paper and introduce the geodesical ray transform. Denote by  $\operatorname{div} X$  the divergence of a vector field  $X \in H^1(T\mathcal{M})$  on  $\mathcal{M}$ , i.e. in local coordinates,

$$\operatorname{div} X = \frac{1}{\sqrt{\det g}} \sum_{i=1}^n \partial_i \left( \sqrt{\det g} \alpha_i \right), \quad X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}. \quad (2.1)$$



If  $X \in H^1(T\mathcal{M})$  the divergence formula reads

$$\int_{\mathcal{M}} \operatorname{div} X \, dv_g^n = \int_{\partial\mathcal{M}} \langle X, \nu \rangle \, d\sigma_g^{n-1} \quad (2.2)$$

and for  $f \in H^1(\mathcal{M})$  Green's formula reads

$$\int_{\mathcal{M}} \operatorname{div} X \, f \, dv_g^n = - \int_{\mathcal{M}} \langle X, \nabla_g f \rangle_g \, dv_g^n + \int_{\partial\mathcal{M}} \langle X, \nu \rangle \, f \, d\sigma_g^{n-1}. \quad (2.3)$$

Then if  $f \in H^1(\mathcal{M})$  and  $w \in H^2(\mathcal{M})$ , the following identity holds

$$\int_{\mathcal{M}} \Delta_g w \, f \, dv_g^n = - \int_{\mathcal{M}} \langle \nabla_g w, \nabla_g f \rangle_g \, dv_g^n + \int_{\partial\mathcal{M}} \partial_\nu w \, f \, d\sigma_g^{n-1}. \quad (2.4)$$

Let  $v \in \mathcal{C}^1(\mathcal{M})$  and  $N$  be a smooth real vector field. The following identity holds true (see [37])

$$\begin{aligned} & \langle \nabla_g v, \nabla_g (\langle N, \nabla_g \bar{v} \rangle_g) \rangle_g \\ &= DN(\nabla_g v, \nabla_g \bar{v}) + \frac{1}{2} \operatorname{div}(|\nabla_g v|_g^2 N) - \frac{1}{2} |\nabla_g v|_g^2 \operatorname{div}(N) \end{aligned} \quad (2.5)$$

where  $D$  is the Levi-Civita connection.

For  $x \in \mathcal{M}$  and  $\theta \in T_x \mathcal{M}$  we denote by  $\gamma_{x,\theta}$  the unique geodesic starting at the point  $x$  in the direction  $\theta$ . We consider

$$\begin{aligned} S\mathcal{M} &= \left\{ (x, \theta) \in T\mathcal{M}; |\theta|_g = 1 \right\}, \\ S^*\mathcal{M} &= \left\{ (x, p) \in T^*\mathcal{M}; |p|_g = 1 \right\} \end{aligned}$$

the sphere bundle and co-sphere bundle of  $\mathcal{M}$ . The exponential map  $\exp_x : T_x \mathcal{M} \rightarrow \mathcal{M}$  is given by

$$\exp_x(v) = \gamma_{x,v}(|v|_g v) = \gamma_{x,v}(rv), \quad r = |v|_g. \quad (2.6)$$

A compact Riemannian manifold  $(\mathcal{M}, g)$  with boundary is called a convex non-trapping manifold, if it satisfies two conditions:

- (a) the boundary  $\partial\mathcal{M}$  is strictly convex, i.e., the second fundamental form of the boundary is positive definite at every boundary point,
- (b) for every point  $x \in \mathcal{M}$  and every vector  $\theta \in T_x \mathcal{M}$ ,  $\theta \neq 0$ , the maximal geodesic  $\gamma_{x,\theta}(t)$  satisfying the initial conditions  $\gamma_{x,\theta}(0) = x$  and  $\dot{\gamma}_{x,\theta}(0) = \theta$  is defined on a finite segment  $[\tau_-(x, \theta), \tau_+(x, \theta)]$ . We recall that a geodesic  $\gamma : [a, b] \rightarrow M$  is maximal if it cannot be extended to a segment  $[a - \varepsilon_1, b + \varepsilon_2]$ , where  $\varepsilon_i \geq 0$  and  $\varepsilon_1 + \varepsilon_2 > 0$ .

The second condition is equivalent to all geodesics having finite length in  $\mathcal{M}$ .

An important subclass of convex non-trapping manifolds are simple manifolds. We say that a compact Riemannian manifold  $(\mathcal{M}, g)$  is simple if it satisfies the following properties

- (a) the boundary is strictly convex,
- (b) there are no conjugate points on any geodesic.

A simple  $n$ -dimensional Riemannian manifold is diffeomorphic to a closed ball in  $\mathbb{R}^n$ , and any pair of points in the manifold are joined by a unique geodesic.

Now, we introduce the submanifolds of inner and outer vectors of  $S\mathcal{M}$

$$\partial_{\pm}S\mathcal{M} = \{(x, \theta) \in S\mathcal{M}, x \in \partial\mathcal{M}, \pm \langle \theta, \nu(x) \rangle < 0\} \quad (2.7)$$

where  $\nu$  is the unit outer normal to the boundary. Note that  $\partial_+S\mathcal{M}$  and  $\partial_-S\mathcal{M}$  are compact manifolds with the same boundary  $S(\partial\mathcal{M})$ , and  $\partial S\mathcal{M} = \partial_+S\mathcal{M} \cup \partial_-S\mathcal{M}$ . For  $(x, \theta) \in \partial_+S\mathcal{M}$ , we denote by  $\gamma_{x,\theta} : [0, \tau_+(x, \theta)] \rightarrow \mathcal{M}$  the maximal geodesic satisfying the initial conditions  $\gamma_{x,\theta}(0) = x$  and  $\dot{\gamma}_{x,\theta}(0) = \theta$ . Let  $\mathcal{C}^\infty(\partial_+S\mathcal{M})$  be the space of smooth functions on the manifold  $\partial_+S\mathcal{M}$ . The ray transform (also called geodesic X-ray transform) on a convex non trapping manifold  $\mathcal{M}$  is the linear operator

$$\mathcal{I} : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathcal{C}^\infty(\partial_+S\mathcal{M}) \quad (2.8)$$

defined by the equality

$$\mathcal{I}f(x, \theta) = \int_0^{\tau_+(x, \theta)} f(\gamma_{x,\theta}(t)) dt. \quad (2.9)$$

The right-hand side of (2.9) is a smooth function on  $\partial_+S\mathcal{M}$  because the integration limit  $\tau_+(x, \theta)$  is a smooth function on  $\partial_+S\mathcal{M}$ , see Lemma 4.1.1 of [32]. The ray transform on a convex non trapping manifold  $\mathcal{M}$  can be extended as a bounded operator

$$\mathcal{I} : H^k(\mathcal{M}) \longrightarrow H^k(\partial_+S\mathcal{M}) \quad (2.10)$$

for every integer  $k \geq 1$ , see Theorem 4.2.1 of [32].

The Riemannian scalar product on  $T_x\mathcal{M}$  induces the volume form on  $S_x\mathcal{M}$ , denoted by  $d\omega_x(\theta)$  and given by

$$d\omega_x(\theta) = \sum_{k=1}^n (-1)^k \theta^k d\theta^1 \wedge \cdots \wedge \widehat{d\theta^k} \wedge \cdots \wedge d\theta^n.$$

We introduce the volume form  $\mathrm{d}v_g^{2n-1}$  on the manifold  $S\mathcal{M}$  by

$$\mathrm{d}v_g^{2n-1}(x, \theta) = |\mathrm{d}\omega_x(\theta) \wedge \mathrm{d}v_g^n|$$

where  $\mathrm{d}v_g^n$  is the Riemannian volume form on  $\mathcal{M}$ . By Liouville's theorem, the form  $\mathrm{d}v_g^{2n-1}$  is preserved by the geodesic flow. The corresponding volume form on the boundary  $\partial S\mathcal{M} = \{(x, \theta) \in S\mathcal{M}, x \in \partial\mathcal{M}\}$  is given by

$$\mathrm{d}\sigma_g^{2n-2} = |\mathrm{d}\omega_x(\theta) \wedge \mathrm{d}\sigma_g^{n-1}|$$

where  $\mathrm{d}\sigma_g^{n-1}$  is the volume form of  $\partial\mathcal{M}$ .

Let  $L_\mu^2(\partial_+ S\mathcal{M})$  be the space of square integrable functions with respect to the measure  $\mu(x, \theta) \mathrm{d}\sigma_g^{2n-2}$  with  $\mu(x, \theta) = |\langle \theta, \nu(x) \rangle|$ . This real Hilbert space is endowed with the scalar product

$$\langle u, v \rangle_{L_\mu^2(\partial_+ S\mathcal{M})} = \int_{\partial_+ S\mathcal{M}} u(x, \theta) v(x, \theta) \mu(x, \theta) \mathrm{d}\sigma_g^{2n-2}. \quad (2.11)$$

The ray transform  $\mathcal{I}$  is a bounded operator from  $L^2(\mathcal{M})$  into  $L_\mu^2(\partial_+ S\mathcal{M})$ . The adjoint  $\mathcal{I}^* : L_\mu^2(\partial_+ S\mathcal{M}) \rightarrow L^2(\mathcal{M})$  is given by

$$\mathcal{I}^* \psi(x) = \int_{S_x \mathcal{M}} \psi^*(x, \theta) \mathrm{d}\omega_x(\theta) \quad (2.12)$$

where  $\psi^*$  is the extension of the function  $\psi$  from  $\partial_+ S\mathcal{M}$  to  $S\mathcal{M}$  constant on every orbit of the geodesic flow, i.e.

$$\psi^*(x, \theta) = \psi(\gamma_{x, \theta}(\tau_+(x, \theta))).$$

Let  $(\mathcal{M}, g)$  be a simple metric, we assume that  $g$  extends smoothly as a simple metric on  $\mathcal{M}_1 \supset \mathcal{M}$ . Then there exist  $C_1 > 0, C_2 > 0$  such that

$$C_1 \|f\|_{L^2(\mathcal{M})} \leq \|\mathcal{I}^* \mathcal{I}(f)\|_{H^1(\mathcal{M}_1)} \leq C_2 \|f\|_{L^2(\mathcal{M})} \quad (2.13)$$

for any  $f \in L^2(\mathcal{M})$ . If  $V$  is an open set of the simple Riemannian manifold  $(\mathcal{M}_1, g)$ , the normal operator  $\mathcal{I}^* \mathcal{I}$  is an elliptic pseudodifferential operator of order  $-1$  on  $V$  whose principal symbol is a multiple of  $|\xi|_g$  (see [34]). Therefore there exists a constant  $C_k > 0$  such that for all  $f \in H^k(V)$  compactly supported in  $V$

$$\|\mathcal{I}^* \mathcal{I}(f)\|_{H^{k+1}(\mathcal{M}_1)} \leq C_k \|f\|_{H^k(V)}. \quad (2.14)$$

### 3 The Cauchy problem for the Schrödinger equation

In this section we will establish existence, uniqueness and continuous dependence on the data of solutions to the Schrödinger equation (1.2) with non-homogenous Dirichlet boundary condition  $f \in H^1((0, T) \times \partial\mathcal{M})$ . We will use the method of transposition, or adjoint isomorphism of equations, and we shall solve the case of non-homogenous Dirichlet boundary conditions under stronger assumptions on the data than those in [4].

Let us first review the classical well-posedness results for the Schrödinger equation with homogenous boundary conditions. After applying the transposition method, we establish Theorem 1.

#### 3.1 Homogenous boundary condition

Let us consider the following initial and homogenous boundary value problem for the Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_g + q(x)) v(t, x) = F(t, x) & \text{in } (0, T) \times \mathcal{M}, \\ v(0, x) = 0 & \text{in } \mathcal{M}, \\ v(t, x) = 0 & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (3.1)$$

Firstly, it is well known that if  $F \in L^1(0, T; L^2(\mathcal{M}))$  then (3.1) admits an unique weak solution

$$v \in \mathcal{C}(0, T; L^2(\mathcal{M})). \quad (3.2)$$

If we multiply both sides of the first equation (3.1) by  $\bar{v}$  and integrate over  $\mathcal{M}$ , we obtain

$$\begin{aligned} & \operatorname{Im} \left[ \int_{\mathcal{M}} i\partial_t v(t) \bar{v} \, dv_g^n - \int_{\mathcal{M}} |\nabla_g v(t, x)|_g^2 + q |v(t, x)|^2 \, dv_g^n \right] \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathcal{M}} |v(t, x)|^2 \, dv_g^n = \operatorname{Im} \int_{\mathcal{M}} \left( F(t, x) \bar{v}(t, x) + q |v(t, x)|^2 \right) \, dv_g^n. \end{aligned} \quad (3.3)$$

take  $\alpha_0(t) = \|v(t)\|_{L^2(\mathcal{M})}$ ,  $t \in (0, T)$ , we get

$$\frac{d}{dt} (\alpha_0^2(t)) \leq C \left( \|F(t, \cdot)\|_{L^2(\mathcal{M})} \alpha_0(t) + \alpha_0^2(t) \right), \quad \forall t \in (0, T), \quad (3.4)$$

which implies that  $\alpha_0'(t) \leq C \left( \|F(t, \cdot)\|_{L^2(\mathcal{M})} + \alpha_0(t) \right)$  and

$$\|v(t)\|_{L^2(\mathcal{M})} \leq C_T \int_0^t \|F(s, \cdot)\|_{L^2(\mathcal{M})} \, ds, \quad \forall t \in (0, T). \quad (3.5)$$

**Lemma 3.1** *Let  $T > 0$  and  $q \in W^{1,\infty}(\mathcal{M})$ . Suppose that  $F \in L^1(0, T; H_0^1(\mathcal{M}))$ . Then the unique solution  $v$  of (3.1) satisfies*

$$v \in \mathcal{C}(0, T; H_0^1(\mathcal{M})). \quad (3.6)$$

*Furthermore there is a constant  $C > 0$  such that for any  $F \in L^1(0, T; H_0^1(\mathcal{M}))$ , we have*

$$\|v(t, \cdot)\|_{H_0^1(\mathcal{M})} \leq C \|F\|_{L^1(0, T; H_0^1(\mathcal{M}))}. \quad (3.7)$$

**Proof .** Using the classical result of existence and uniqueness of weak solutions in Cazenave and Haraux [14] (set for abstract evolution equations), we obtain

$$v \in \mathcal{C}(0, T; H_0^1(\mathcal{M})). \quad (3.8)$$

Multiplying the first equation of (3.1) by  $\Delta_g \bar{v}$  and using Green's formula, we get

$$\begin{aligned} \operatorname{Im} \left[ \int_{\mathcal{M}} i \partial_t v(t) \Delta_g \bar{v} \, dv_g^n - \int_{\mathcal{M}} |\Delta_g v(t)|^2 + qv \Delta_g \bar{v} \, dv_g^n \right] \\ = -\frac{1}{2} \frac{d}{dt} \int_{\mathcal{M}} |\nabla_g v(t)|^2 \, dv_g^n + \operatorname{Im} \int_{\mathcal{M}} \langle \nabla_g(qv), \nabla_g \bar{v} \rangle_g \, dv_g^n \\ = \operatorname{Im} \int_{\mathcal{M}} \langle \nabla_g F(t, x), \nabla_g \overline{v(t)} \rangle \, dv_g^n. \end{aligned} \quad (3.9)$$

Let  $\alpha_1(t) = \|\nabla_g v(t)\|_{L^2(\mathcal{M})}$ ,  $t \in (0, T)$ . Then, by (3.9), we have

$$\frac{d}{dt} (\alpha_1^2(t)) \leq C \left( \|F(t, \cdot)\|_{H_0^1(\mathcal{M})} \alpha_1(t) + \alpha_1^2(t) \right), \quad \forall t \in (0, T), \quad (3.10)$$

which implies that  $\alpha_1'(t) \leq C \left( \|F(t, \cdot)\|_{H_0^1(\mathcal{M})} + \alpha_1(t) \right)$  and by Gronwall's lemma we find

$$\|v(t)\|_{H_0^1(\mathcal{M})} \leq C_T \int_0^T \|F(t, \cdot)\|_{H_0^1(\mathcal{M})} \, dt, \quad \forall t \in (0, T). \quad (3.11)$$

The proof of (3.7) is complete.  $\square$

**Lemma 3.2** *Let  $T > 0$  and  $q \in L^\infty(\mathcal{M})$ . Suppose that  $F \in W^{1,1}(0, T; L^2(\mathcal{M}))$  such that  $F(0, \cdot) \equiv 0$ . Then the unique solution  $v$  of (3.1) satisfies*

$$v \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})). \quad (3.12)$$

*Furthermore there is a constant  $C > 0$  such that for any  $\eta > 0$  small, we have*

$$\|v(t, \cdot)\|_{H_0^1(\mathcal{M})} \leq C \left( \eta \|\partial_t F\|_{L^1(0, T; L^2(\mathcal{M}))} + \eta^{-1} \|F\|_{L^1(0, T; L^2(\mathcal{M}))} \right). \quad (3.13)$$

**Proof .** If we consider the equation satisfied by  $\partial_t v$ , (3.2) provides the following regularity

$$v \in \mathcal{C}^1(0, T; L^2(\mathcal{M})).$$

Furthermore by (3.5), there is a constant  $C > 0$  such that the following estimate holds true

$$\|\partial_t v(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \int_0^T \|\partial_t F(t, \cdot)\|_{L^2(\mathcal{M})} dt \quad \forall t \in (0, T). \quad (3.14)$$

Then, by (3.1), we see that  $\Delta_g v = -i\partial_t v + F \in \mathcal{C}(0, T; L^2(\mathcal{M}))$  and therefore

$$v \in \mathcal{C}(0, T; H^2(\mathcal{M})).$$

This complete the proof of (3.12).

Next, multiplying the first equation of (3.1) by  $\bar{v}$  and integrating by parts, we obtain

$$\begin{aligned} \operatorname{Re} \left[ \int_{\mathcal{M}} i\partial_t v(t, x) \bar{v}(t, x) dv_g^n - \int_{\mathcal{M}} (|\nabla_g v(t)|^2 - q|v|^2) dv_g^n \right] \\ = \operatorname{Re} \int_{\mathcal{M}} F(t, x) \bar{v}(t, x) dv_g^n \\ = \operatorname{Re} \int_{\mathcal{M}} \left( \int_0^t \partial_t F(s, x) ds \right) \bar{v}(t, x) dv_g^n. \end{aligned} \quad (3.15)$$

Take now  $\alpha_1(t) = \|\nabla_g v(t)\|_{L^2(\mathcal{M})}$ . Then there exists a constant  $C > 0$  such that the following estimate holds true

$$\begin{aligned} \alpha_1^2(t) \leq C \left[ \|\partial_t v(t, \cdot)\|_{L^2(\mathcal{M})} \|v(t, \cdot)\|_{L^2(\mathcal{M})} + \|v(t)\|_{L^2(\mathcal{M})}^2 \right. \\ \left. + \int_0^T \int_{\mathcal{M}} |v(t, x) \partial_t F(s, x)| dv_g^n ds \right]. \end{aligned} \quad (3.16)$$

Using (3.14) and (3.5), we get

$$\alpha_1^2(t) \leq C \left[ \|\partial_t F\|_{L^1(0, T; L^2(\mathcal{M}))} \|F\|_{L^1(0, T; L^2(\mathcal{M}))} + \|F\|_{L^1(0, T; L^2(\mathcal{M}))}^2 \right]. \quad (3.17)$$

Thus, we deduce (3.13), and this concludes the proof of Lemma 3.2.  $\square$

**Lemma 3.3** *Let  $T > 0$ ,  $q \in W^{1, \infty}(\mathcal{M})$  be given and let  $\mathcal{H} = L^1(0, T; H_0^1(\mathcal{M}))$  or  $\mathcal{H} = H_0^1(0, T; L^2(\mathcal{M}))$ . Then the mapping  $F \mapsto \partial_\nu v$  where  $v$  is the unique solution to (3.1) is linear and continuous from  $\mathcal{H}$  to  $L^2((0, T) \times \partial\mathcal{M})$ . Furthermore, there is a constant  $C > 0$  such that*

$$\|\partial_\nu v\|_{L^2((0, T) \times \partial\mathcal{M})} \leq C \|F\|_{\mathcal{H}}. \quad (3.18)$$

**Proof .** Let  $N$  be a  $\mathcal{C}^2$  vector field on  $\overline{\mathcal{M}}$  such that

$$N(x) = \nu(x), \quad x \in \partial\mathcal{M}; \quad |N(x)|_{\mathbf{g}} \leq 1, \quad x \in \mathcal{M}. \quad (3.19)$$

Multiply both sides of the first equation in (3.1) by  $\langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}}$  and integrate over  $(0, T) \times \mathcal{M}$ , this gives

$$\begin{aligned} & \int_0^T \int_{\mathcal{M}} F(t, x) \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ &= \int_0^T \int_{\mathcal{M}} i \partial_t v \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt + \int_0^T \int_{\mathcal{M}} \Delta_{\mathbf{g}} \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ & \quad + \int_0^T \int_{\mathcal{M}} q(x) v \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (3.20)$$

Consider the first term on left side of (3.20); integrating by parts with respect  $t$ , we get

$$\begin{aligned} I_1 &= \int_0^T \int_{\mathcal{M}} i \partial_t v \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ &= i \left[ \int_{\mathcal{M}} v \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \right]_0^T - i \int_0^T \int_{\mathcal{M}} v \langle N, \nabla_{\mathbf{g}} \partial_t \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ &= i \int_{\mathcal{M}} v(T, x) \langle N, \nabla_{\mathbf{g}} \bar{v}(T, x) \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n - i \int_0^T \int_{\mathcal{M}} \langle N, \nabla_{\mathbf{g}} (v \partial_t \bar{v}) \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ & \quad + i \int_0^T \int_{\mathcal{M}} \partial_t \bar{v} \langle N, \nabla_{\mathbf{g}} v \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt. \end{aligned} \quad (3.21)$$

Then, by (2.3), we obtain

$$\begin{aligned} & \operatorname{Re} \left[ \int_0^T \int_{\mathcal{M}} i \partial_t v \langle N, \nabla_{\mathbf{g}} \bar{v} \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \right] \\ &= i \int_{\mathcal{M}} v(T, x) \langle N, \nabla_{\mathbf{g}} \bar{v}(T, x) \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n + i \int_0^T \int_{\mathcal{M}} \operatorname{div}(N) v \partial_t \bar{v} \, dv_{\mathbf{g}}^n \, dt \\ & \quad - i \left[ \int_0^T \int_{\partial\mathcal{M}} v \partial_t \bar{v} \, d\sigma_{\mathbf{g}}^{n-1} \, dt \right] \\ &= i \int_{\mathcal{M}} v(T, x) \langle N, \nabla_{\mathbf{g}} \bar{v}(T, x) \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n + \int_0^T \int_{\mathcal{M}} \langle \nabla_{\mathbf{g}} \bar{v}, \nabla_{\mathbf{g}} (\operatorname{div}(N) v) \rangle_{\mathbf{g}} \, dv_{\mathbf{g}}^n \, dt \\ & \quad + \int_0^T \int_{\mathcal{M}} F \operatorname{div}(N) v \, dv_{\mathbf{g}}^n \, dt - \int_0^T \int_{\mathcal{M}} q \operatorname{div}(N) |v|^2 \, dv_{\mathbf{g}}^n \, dt \end{aligned}$$

$$- \left[ i \int_0^T \int_{\partial \mathcal{M}} v \partial_t \bar{v} d\sigma_g^{n-1} dt + \int_0^T \int_{\partial \mathcal{M}} \partial_\nu \bar{v} v \operatorname{div}(N) d\sigma_g^{n-1} dt \right].$$

The last term vanishes, using (3.13) or (3.7), we conclude that

$$|\operatorname{Re} I_1| \leq C \|F\|_{\mathcal{H}}^2. \quad (3.22)$$

On the other hand, by Green's theorem, we get

$$\begin{aligned} I_2 &= \int_0^T \int_{\mathcal{M}} \Delta_g v \langle N, \nabla_g \bar{v} \rangle_g dv_g^n dt \\ &= - \int_0^T \int_{\mathcal{M}} \langle \nabla_g v, \nabla_g (\langle N, \nabla_g \bar{v} \rangle_g) \rangle_g dv_g^n dt + \int_0^T \int_{\partial \mathcal{M}} |\partial_\nu v|^2 d\sigma_g^{n-1} dt. \end{aligned}$$

Thus by (2.5), we deduce

$$\begin{aligned} I_2 &= \int_0^T \int_{\partial \mathcal{M}} |\partial_\nu v|^2 d\sigma_g^{n-1} dt - \frac{1}{2} \int_0^T \int_{\partial \mathcal{M}} |\nabla_g v|^2 d\sigma_g^{n-1} dt \\ &\quad + \int_0^T \int_{\mathcal{M}} D_g N(\nabla_g v, \nabla_g \bar{v}) dv_g^n dt - \frac{1}{2} \int_0^T \int_{\mathcal{M}} |\nabla_g v|_g^2 \operatorname{div}(N) dv_g^n dt. \end{aligned}$$

Using the fact

$$|\nabla_g v|_g^2 = |\partial_\nu v|^2 + |\nabla_\tau v|_g^2 = |\partial_\nu v|^2, \quad x \in \partial \mathcal{M}$$

where  $\nabla_\tau$  is the tangential gradient on  $\partial \mathcal{M}$ , we get

$$\begin{aligned} \operatorname{Re} I_2 &= \frac{1}{2} \int_0^T \int_{\partial \mathcal{M}} |\partial_\nu v|^2 d\sigma_g^{n-1} dt + \int_0^T \int_{\mathcal{M}} D_g N(\nabla_g v, \nabla_g \bar{v}) dv_g^n dt \\ &\quad - \frac{1}{2} \int_0^T \int_{\mathcal{M}} |\nabla_g v|_g^2 \operatorname{div}(N) dv_g^n dt. \end{aligned} \quad (3.23)$$

Finally by Lemma 3.1 and 3.2, we have

$$|\operatorname{Re} I_3| \leq \|F\|_{\mathcal{H}}^2. \quad (3.24)$$

Collecting (3.24), (3.23), (3.22) and (3.20), we obtain

$$\int_0^T \int_{\partial \mathcal{M}} |\partial_\nu v|^2 d\sigma_g^{n-1} dt \leq C \|F\|_{\mathcal{H}}^2. \quad (3.25)$$

This completes the proof of (3.18).  $\square$



### 3.2 Non-homogenous boundary condition

We now turn to the non-homogenous Schrödinger problem (1.2). Let

$$\mathcal{H} = L^1(0, T; H_0^1(\mathcal{M})) \text{ or } \mathcal{H} = H_0^1(0, T; L^2(\mathcal{M})).$$

By  $(\cdot, \cdot)_{\mathcal{H}', \mathcal{H}}$ , we denote the dual pairing between  $\mathcal{H}'$  and  $\mathcal{H}$ .

**Definition 2** Let  $T > 0$ ,  $q \in W^{1, \infty}(\mathcal{M})$  and  $f \in L^2((0, T) \times \partial\mathcal{M})$ , we say that  $u \in \mathcal{H}'$  is a solution of (1.2) in the transposition sense if for any  $F \in \mathcal{H}$  we have

$$(u, F)_{\mathcal{H}', \mathcal{H}} = \int_0^T \int_{\partial\mathcal{M}} f(t, x) \partial_\nu \bar{v}(t, x) \, d\sigma_g^{n-1} \, dt \quad (3.26)$$

where  $v = v(t, x)$  is the solution of the homogenous boundary value problem

$$\begin{cases} (i\partial_t + \Delta_g + q(x)) v(t, x) = F(t, x) & \text{in } (0, T) \times \mathcal{M}, \\ v(T, x) = 0 & \text{in } \mathcal{M}, \\ v(t, x) = 0 & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (3.27)$$

One gets the following lemma.

**Lemma 3.4** Let  $f \in L^2((0, T) \times \partial\mathcal{M})$ . There exists a unique solution

$$u \in \mathcal{C}(0, T; H^{-1}(\mathcal{M})) \cap H^{-1}(0, T; L^2(\mathcal{M})) \quad (3.28)$$

defined by transposition, of the problem

$$\begin{cases} (i\partial_t + \Delta_g + q(x)) u(t, x) = 0 & \text{in } (0, T) \times \mathcal{M}, \\ u(0, x) = 0 & \text{in } \mathcal{M}, \\ u(t, x) = f(t, x) & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (3.29)$$

Furthermore, there is a constant  $C > 0$  such that

$$\|u\|_{\mathcal{C}(0, T; H^{-1}(\mathcal{M}))} + \|u\|_{H^{-1}(0, T; L^2(\mathcal{M}))} \leq C \|f\|_{L^2((0, T) \times \partial\mathcal{M})}. \quad (3.30)$$

**Proof .** Let  $F \in \mathcal{H} = L^1(0, T; H_0^1(\mathcal{M}))$  or  $\mathcal{H} = H_0^1(0, T; L^2(\mathcal{M}))$ . Let  $v \in \mathcal{C}(0, T; H_0^1(\mathcal{M}))$  solution of the backward boundary value problem for the Schrödinger equation (3.27). By Lemma 3.3 the mapping  $F \mapsto \frac{\partial v}{\partial \nu}$  is linear and continuous from  $\mathcal{H}$  to  $L^2((0, T) \times \mathcal{M})$  and there exists  $C > 0$  such that

$$\|v\|_{\mathcal{C}(0, T; H_0^1(\mathcal{M}))} \leq C \|F\|_{\mathcal{H}} \quad (3.31)$$

and

$$\|\partial_\nu v\|_{L^2((0,T)\times\partial\mathcal{M})} \leq C \|F\|_{\mathcal{H}}. \quad (3.32)$$

We define a linear functional  $\ell$  on the linear space  $\mathcal{H}$  as follows:

$$\ell(F) = \langle f, \partial_\nu v \rangle_0$$

where  $v$  solves (3.27). By (3.32), we obtain

$$|\ell(F)| \leq \|f\|_{L^2((0,T)\times\partial\mathcal{M})} \|F\|_{\mathcal{H}}.$$

It is known that any linear bounded functional on the space  $\mathcal{H}$  can be written as

$$\ell(F) = (u, F)_{\mathcal{H}', \mathcal{H}}$$

where  $u$  is some element from the space  $\mathcal{H}'$ . Thus the system (3.29) admits a solution  $u \in \mathcal{H}'$  in the transposition sense, which satisfies

$$\|u\|_{\mathcal{H}'} \leq C \|f\|_{L^2((0,T)\times\partial\mathcal{M})}.$$

This completes the proof of the Lemma.  $\square$

In what follows, we will need the following estimate for non-homogenous elliptic boundary value problem.

**Lemma 3.5** *Let  $\psi \in H^{-1}(\mathcal{M})$  and  $\phi \in H^1(\partial\mathcal{M})$ . Let  $w \in H^1(\mathcal{M})$  the unique solution of the following boundary value problem*

$$\begin{cases} \Delta_g w(x) = \psi(x) & \text{in } \mathcal{M}, \\ w(x) = \phi & \text{on } \partial\mathcal{M}, \end{cases} \quad (3.33)$$

*then the following estimate holds true*

$$\|w\|_{H^1(\mathcal{M})} \leq C \left( \|\psi\|_{H^{-1}(\mathcal{M})} + \|\phi\|_{H^1(\partial\mathcal{M})} \right). \quad (3.34)$$

**Proof .** We decompose the solution  $w$  of (3.33) as  $w = w_1 + w_2$  with  $w_1$  and  $w_2$ , respectively, solution of

$$\begin{cases} \Delta_g w_1(x) = \psi(x) & \text{in } \mathcal{M}, \\ w_1(x) = 0 & \text{on } \partial\mathcal{M}, \end{cases} ; \quad \begin{cases} \Delta_g w_2(x) = 0 & \text{in } \mathcal{M}, \\ w_2(x) = \phi & \text{on } \partial\mathcal{M}, \end{cases} \quad (3.35)$$

Since  $-\Delta_g$  is an isomorphism from  $H_0^1(\mathcal{M})$  to  $H^{-1}(\mathcal{M})$ , we have

$$\|w_1\|_{H^1(\mathcal{M})} \leq C \|\psi\|_{H^{-1}(\mathcal{M})}. \quad (3.36)$$

Next, it is well known that (see [28])

$$\|w_2\|_{L^2(\mathcal{M})} \leq C \|\phi\|_{L^2(\partial\mathcal{M})}. \quad (3.37)$$

Now, we shall show that

$$\|w_2\|_{H^1(\mathcal{M})} \leq C \|\phi\|_{H^1(\partial\mathcal{M})}. \quad (3.38)$$

Indeed, let  $h \in H^1(\partial\mathcal{M})$  and  $\theta \in C_0^\infty(0, T)$ ,  $\theta \geq 0$ . Let  $v$  solve the following initial boundary value problem for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g) v(t, x) = 0 & \text{in } (0, T) \times \mathcal{M}, \\ v(0, x) = \partial_t v(0, x) = 0 & \text{in } \mathcal{M}, \\ v(t, x) = h(x)\theta(t) & \text{on } (0, T) \times \partial\mathcal{M}. \end{cases} \quad (3.39)$$

Then we have (see [25])

$$v \in \mathcal{C}(0, T; H^1(\mathcal{M})) \cap \mathcal{C}^1(0, T; L^2(\mathcal{M})).$$

Furthermore there exist  $C > 0$  such that

$$\begin{aligned} \|v\|_{\mathcal{C}(0, T; H^1(\mathcal{M}))} + \|v\|_{\mathcal{C}^1(0, T; L^2(\mathcal{M}))} + \|\partial_\nu v\|_{L^2((0, T) \times \partial\mathcal{M})} \\ \leq C \|h\|_{H^1(\partial\mathcal{M})}. \end{aligned} \quad (3.40)$$

Multiplying both sides of (3.39) by  $w_2$  and integrating over  $(0, T) \times \mathcal{M}$ , we get

$$\begin{aligned} 0 &= \int_0^T \int_{\mathcal{M}} (\partial_t^2 v - \Delta_g v) w_2(x) \, dv_g^n \, dt \\ &= \int_{\mathcal{M}} \partial_t v(T, x) w_2(x) \, dv_g^n - \int_0^T \int_{\partial\mathcal{M}} \partial_\nu v \phi(x) \, d\sigma_g^{n-1} \, dt \\ &\quad + \int_0^T \theta(t) \, dt \int_{\partial\mathcal{M}} h(x) \partial_\nu w_2 \, d\sigma_g^{n-1}. \end{aligned} \quad (3.41)$$

Then, by (3.40) and (3.37), one gets

$$\begin{aligned} \left| \int_{\partial\mathcal{M}} h(x) \partial_\nu w_2 \, d\sigma_g^{n-1} \right| &\leq C \left( \|\partial_\nu v\|_{L^2((0, T) \times \partial\mathcal{M})} \|\phi\|_{L^2(\partial\mathcal{M})} \right. \\ &\quad \left. + \|w_2\|_{L^2(\mathcal{M})} \|v\|_{\mathcal{C}^1(0, T; L^2(\mathcal{M}))} \right) \\ &\leq C \|\phi\|_{L^2(\partial\mathcal{M})} \|h\|_{H^1(\partial\mathcal{M})}. \end{aligned} \quad (3.42)$$

This implies

$$\|\partial_\nu w_2\|_{H^{-1}(\partial\mathcal{M})} \leq C \|\phi\|_{L^2(\partial\mathcal{M})}. \quad (3.43)$$

Furthermore, Green's formula yields

$$\int_{\mathcal{M}} |\nabla_g w_2|^2 \, dv_g^n \leq \|\partial_\nu w_2\|_{H^{-1}(\partial\mathcal{M})} \|\phi\|_{H^1(\partial\mathcal{M})} \leq C \|\phi\|_{H^1(\partial\mathcal{M})}^2. \quad (3.44)$$

From (3.44) and (3.37), we get

$$\|w_2\|_{H^1(\mathcal{M})} \leq C \|\phi\|_{H^1(\partial\mathcal{M})}. \quad (3.45)$$

Both (3.45) and (3.36) yield likewise

$$\|w\|_{H^1(\mathcal{M})} \leq C (\|\psi\|_{H^{-1}(\mathcal{M})} + \|\phi\|_{H^1(\partial\mathcal{M})}). \quad (3.46)$$

This completes the proof of (3.34).  $\square$

### 3.3 Proof of Theorem 1

We proceed to prove Theorem 1. Let  $f \in H^1((0, T) \times \partial\mathcal{M})$  and  $u$  solve (1.2). Put  $u' = \partial_t u$ , then

$$\begin{cases} (i\partial_t + \Delta_g + q(x)) u'(t, x) = 0 & \text{in } (0, T) \times \mathcal{M}, \\ u'(0, x) = 0 & \text{in } \mathcal{M}, \\ u'(t, x) = f'(t, x) & \text{on } (0, T) \times \partial\mathcal{M}, \end{cases} \quad (3.47)$$

Since  $f' \in L^2((0, T) \times \partial\mathcal{M})$ , by lemma 3.4, we get

$$u' \in \mathcal{C}(0, T; H^{-1}(\mathcal{M})) \cap H^{-1}(0, T; L^2(\mathcal{M})). \quad (3.48)$$

Furthermore there is a constant  $C > 0$  such that

$$\|u'\|_{\mathcal{C}(0, T; H^{-1}(\mathcal{M}))} + \|u'\|_{H^{-1}(0, T; L^2(\mathcal{M}))} \leq C \|f\|_{H^1((0, T) \times \partial\mathcal{M})}. \quad (3.49)$$

Thus (3.48) implies the following regularity for  $u$

$$\begin{aligned} u &\in \mathcal{C}^1(0, T; H^{-1}(\mathcal{M})) \cap \mathcal{C}(0, T; L^2(\mathcal{M})), \\ \Delta_g u &\in \mathcal{C}(0, T; H^{-1}(\mathcal{M})) \cap H^{-1}(0, T; L^2(\mathcal{M})). \end{aligned}$$

Since  $f(t, \cdot) \in H^1(\partial\mathcal{M})$ , by the elliptic regularity, we get

$$u \in \mathcal{C}(0, T; H^1(\mathcal{M})) \cap \mathcal{C}^1(0, T; H^{-1}(\mathcal{M})).$$

Moreover there exists  $C > 0$  such that the following estimates hold true

$$\begin{aligned} \|u\|_{\mathcal{C}^1(0, T; H^{-1}(\mathcal{M}))} &\leq C \|f\|_{H^1((0, T) \times \partial\mathcal{M})}, \\ \|\Delta_g u\|_{\mathcal{C}(0, T; H^{-1}(\mathcal{M}))} &\leq C \|f\|_{H^1((0, T) \times \partial\mathcal{M})}. \end{aligned} \quad (3.50)$$

Using Lemma 3.5, we find

$$\begin{aligned} \|u\|_{\mathcal{C}^1(0, T; H^{-1}(\mathcal{M}))} &\leq C \|f\|_{H^1((0, T) \times \partial\mathcal{M})}, \\ \|u\|_{\mathcal{C}(0, T; H^1(\mathcal{M}))} &\leq C \|f\|_{H^1((0, T) \times \partial\mathcal{M})}. \end{aligned} \quad (3.51)$$

The proof of (1.8) is as in Lemma 3.3. If one multiplies (1.2) by  $\langle N, \nabla_g \bar{u} \rangle_g$ , the arguments leading to (3.20) give now

$$\begin{aligned} 0 &= \int_0^T \int_{\mathcal{M}} i \partial_t u \langle N, \nabla_g \bar{u} \rangle_g \, dv_g^n \, dt + \int_0^T \int_{\mathcal{M}} \Delta_g u \langle N, \nabla_g \bar{u} \rangle_g \, dv_g^n \, dt \\ &\quad + \int_0^T \int_{\mathcal{M}} q(x) u \langle N, \nabla_g \bar{u} \rangle_g \, dv_g^n \, dt = I'_1 + I'_2 + I'_3, \end{aligned} \quad (3.52)$$

with

$$|\operatorname{Re} I'_1| \leq C_\varepsilon \|f\|_{H^1((0, T) \times \partial\mathcal{M})}^2 + \varepsilon \|\partial_\nu u\|_{L^2((0, T) \times \partial\mathcal{M})}^2, \quad (3.53)$$

where we have used (3.51) instead of (3.13)-(3.7). Furthermore, we derive from Green's formula

$$\begin{aligned} \operatorname{Re} I'_2 &= \frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\partial_\nu u|^2 \, d\sigma_g^{n-1} \, dt + \int_0^T \int_{\mathcal{M}} D_g N(\nabla_g u, \nabla_g \bar{u}) \, dv_g^n \, dt \\ &\quad - \frac{1}{2} \int_0^T \int_{\mathcal{M}} |\nabla_g u|_g^2 \operatorname{div}(N) \, dv_g^n \, dt - \frac{1}{2} \int_0^T \int_{\partial\mathcal{M}} |\nabla_\tau f|^2 \, d\sigma_g^{n-1} \, dt. \end{aligned} \quad (3.54)$$

This together with

$$|\operatorname{Re} I'_3| \leq \|f\|_{H^1((0, T) \times \partial\mathcal{M})}^2 \quad (3.55)$$

and (3.55), (3.54) and (3.53) imply

$$\|\partial_\nu u\|_{L^2((0, T) \times \partial\mathcal{M})} \leq C \|f\|_{H^1((0, T) \times \partial\mathcal{M})}, \quad (3.56)$$

where we have used (3.51) again. The proof of Theorem 1 is now complete.

## 4 Geometrical optics solutions of the Schrödinger equation

We now proceed to the construction of geometrical optics solutions to the Schrödinger equation. We extend the manifold  $(\mathcal{M}, g)$  into a simple manifold  $\mathcal{M}_2 \ni \mathcal{M}$  and consider a simple manifold  $(\mathcal{M}_1, g)$  such that  $\mathcal{M}_2 \ni \mathcal{M}_1$ . The potentials  $q_1, q_2$  may also be extended to  $\mathcal{M}_2$  and their  $H^1(\mathcal{M}_1)$  norms may be bounded by  $M_0$ . Since  $q_1$  and  $q_2$  coincide on the boundary, their extension outside  $\mathcal{M}$  can be taken the same so that  $q_1 = q_2$  in  $\mathcal{M}_2 \setminus \mathcal{M}_1$ .

Our construction here is a modification of a similar result in [11], which dealt with the situation of the wave equation.

We suppose, for a moment, that is able to find a function  $\psi \in \mathcal{C}^2(\mathcal{M})$  which satisfies the eikonal equation

$$|\nabla_g \psi|_g^2 = \sum_{i,j=1}^n g^{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_j} = 1, \quad \forall x \in \mathcal{M}_2 \quad (4.1)$$

and assume that there exist a function  $a \in H^1(\mathbb{R}, H^2(\mathcal{M}))$  which solves the transport equation

$$\frac{\partial a}{\partial t} + \sum_{j,k=1}^n g^{jk} \frac{\partial \psi}{\partial x_j} \frac{\partial a}{\partial x_k} + \frac{1}{2} (\Delta_g \psi) a = 0, \quad \forall t \in \mathbb{R}, x \in \mathcal{M} \quad (4.2)$$

with initial or final data

$$a(t, x) = 0, \quad \forall x \in \mathcal{M}, \quad \text{and } t \leq 0, \text{ or } t \geq T_0. \quad (4.3)$$

We also introduce the norm  $\|\cdot\|_*$  given by

$$\|a\|_* = \|a\|_{H^1(0, T_0; H^2(\mathcal{M}))}. \quad (4.4)$$

**Lemma 4.1** *Let  $q \in L^\infty(\mathcal{M})$ . Then the following Schrödinger equation*

$$\begin{aligned} (i\partial_t + \Delta_g + q(x))u &= 0, \quad \text{in } \mathcal{M}_T := (0, T) \times \mathcal{M}, \\ u(\kappa, x) &= 0, \quad \kappa = 0, \text{ or } T \end{aligned}$$

*has a solution of the form*

$$u(t, x) = a(2\lambda t, x) e^{i\lambda(\psi(x) - \lambda t)} + v_\lambda(t, x), \quad (4.5)$$

such that

$$u \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M})) \quad (4.6)$$

where  $v_\lambda(t, x)$  satisfies

$$\begin{aligned} v_\lambda(t, x) &= 0, \quad \forall (t, x) \in (0, T) \times \partial\mathcal{M}, \\ v_\lambda(\kappa, x) &= 0, \quad x \in \mathcal{M}, \quad \kappa = 0 \text{ or } T. \end{aligned}$$

Furthermore, there exist  $C > 0$  such that, for any  $\lambda > 0$  the following estimates hold true.

$$\|v_\lambda(t, \cdot)\|_{H^k(\mathcal{M})} \leq C\lambda^{k-1} \|a\|_*, \quad k = 0, 1. \quad (4.7)$$

The constant  $C$  depends only on  $T$  and  $\mathcal{M}$  (that is  $C$  does not depend on  $a$  and  $\lambda$ ).

**Proof .** Let us consider

$$k(t, x) = -(i\partial_t + \Delta_g + q) \left( a(2\lambda t, x) e^{i\lambda(\psi - \lambda t)} \right). \quad (4.8)$$

Let  $v$  solve the following homogenous boundary value problem

$$\begin{cases} (i\partial_t + \Delta_g + q) v(t, x) = k(t, x) & \text{in } (0, T) \times \mathcal{M}, \\ v(\kappa, x) = 0, & \text{in } \mathcal{M}, \tau = 0, \text{ or } T \\ v(t, x) = 0 & \text{on } (0, T) \times \partial\mathcal{M}, \end{cases} \quad (4.9)$$

To prove our Lemma it would be enough to show that  $v$  satisfies the estimates (4.7). We shall prove the estimate for  $\kappa = 0$ , and the  $\kappa = T$  case may be handled in a similar fashion. By a simple computation, we have

$$\begin{aligned} -k(t, x) &= e^{i\lambda(\psi(x) - \lambda t)} (\Delta_g + q(x)) (a(2\lambda t, x)) \\ &\quad + 2i\lambda e^{i\lambda(\psi(x) - \lambda t)} \left( \partial_t a + \sum_{j,k=1}^n g^{jk} \frac{\partial \psi}{\partial x_j} \frac{\partial a}{\partial x_k} + \frac{a}{2} \Delta_g \psi \right) (2\lambda t, x) \\ &\quad + \lambda^2 a(2\lambda t, x) e^{i\lambda(\psi(x) - \lambda t)} \left( 1 - \sum_{j,k=1}^n g^{jk} \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_k} \right). \end{aligned} \quad (4.10)$$

Taking into account (4.1) and (4.2), the right-hand side of (4.10) becomes

$$\begin{aligned} k(t, x) &= -e^{i\lambda(\psi(x) - \lambda t)} (\Delta_g + q) (a(2\lambda t, x)) \\ &\equiv -e^{i\lambda(\psi(x) - \lambda t)} k_0(2\lambda t, x). \end{aligned} \quad (4.11)$$

Since  $k_0 \in H_0^1(0, T; L^2(\mathcal{M}))$ , by Lemma 3.2, we find

$$v_\lambda \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})). \quad (4.12)$$

Furthermore, there is a constant  $C > 0$ , such that

$$\begin{aligned} \|v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} &\leq C \int_0^T \|k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} dt \\ &\leq \frac{C}{\lambda} \int_{\mathbb{R}} \|k_0(s, \cdot)\|_{L^2(\mathcal{M})} ds \\ &\leq \frac{C}{\lambda} \|a\|_*. \end{aligned} \quad (4.13)$$

Moreover, we have

$$\begin{aligned} \|\nabla v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} &\leq C\eta \int_0^T \left( \lambda^2 \|k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \lambda \|\partial_t k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right) dt \\ &\quad + \eta^{-1} \int_0^T \|k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} dt. \end{aligned} \quad (4.14)$$

Finally, choosing  $\eta = \lambda^{-1}$ , we obtain

$$\begin{aligned} \|\nabla v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} &\leq C \left( \int_{\mathbb{R}} \|k_0(s, \cdot)\|_{L^2(\mathcal{M})} ds + \int_{\mathbb{R}} \|\partial_t k_0(s, \cdot)\|_{L^2(\mathcal{M})} ds \right) \\ &\leq C \|a\|_*. \end{aligned} \quad (4.15)$$

Combining (4.15) and (4.13), we immediately deduce the estimate (4.7).  $\square$

We will now construct the phase function  $\psi$  solution to the eikonal equation (4.1) and the amplitude  $a$  solution to the transport equation (4.2).

Let  $y \in \partial\mathcal{M}_1$ . Denote points in  $\mathcal{M}_1$  by  $(r, \theta)$  where  $(r, \theta)$  are polar normal coordinates in  $\mathcal{M}_1$  with center  $y$ . That is  $x = \exp_y(r\theta)$  where  $r > 0$  and

$$\theta \in S_y\mathcal{M}_1 = \left\{ \xi \in T_y\mathcal{M}_1, |\xi|_g = 1 \right\}.$$

In these coordinates (which depend on the choice of  $y$ ) the metric takes the form

$$\tilde{g}(r, \theta) = dr^2 + g_0(r, \theta)$$



where  $g_0(r, \theta)$  is a smooth positive definite metric. For any function  $u$  compactly supported in  $\mathcal{M}$ , we set for  $r > 0$  and  $\theta \in S_y \mathcal{M}_1$

$$\tilde{u}(r, \theta) = u(\exp_y(r\theta))$$

where we have extended  $u$  by 0 outside  $\mathcal{M}$ . An explicit solution to the eikonal equation (4.1) is the geodesic distance function to  $y \in \partial \mathcal{M}_1$

$$\psi(x) = d_g(x, y). \quad (4.16)$$

By the simplicity assumption, since  $y \in \mathcal{M}_2 \setminus \overline{\mathcal{M}}$ , we have  $\psi \in C^\infty(\mathcal{M})$  and

$$\tilde{\psi}(r, \theta) = r = d_g(x, y). \quad (4.17)$$

The next step is to solve the transport equation (4.2). Recall that if  $f(r)$  is any function of the geodesic distance  $r$ , then

$$\Delta_{\tilde{g}} f(r) = f''(r) + \frac{\alpha^{-1}}{2} \frac{\partial \alpha}{\partial r} f'(r). \quad (4.18)$$

Here  $\alpha = \alpha(r, \theta)$  denotes the square of the volume element in geodesic polar coordinates. The transport equation (4.2) becomes

$$\frac{\partial \tilde{a}}{\partial t} + \frac{\partial \tilde{\psi}}{\partial r} \frac{\partial \tilde{a}}{\partial r} + \frac{1}{4} \tilde{a} \alpha^{-1} \frac{\partial \alpha}{\partial r} \frac{\partial \tilde{\psi}}{\partial r} = 0. \quad (4.19)$$

Thus  $\tilde{a}$  satisfy

$$\frac{\partial \tilde{a}}{\partial t} + \frac{\partial \tilde{a}}{\partial r} + \frac{1}{4} \tilde{a} \alpha^{-1} \frac{\partial \alpha}{\partial r} = 0. \quad (4.20)$$

Let  $\phi \in C_0^\infty(\mathbb{R})$  and  $b \in H^2(\partial_+ S\mathcal{M})$ . Let us write  $\tilde{a}$  in the form

$$\tilde{a}(t, r, \theta) = \alpha^{-1/4} \phi(t - r) b(y, \theta). \quad (4.21)$$

Direct computations yields

$$\frac{\partial \tilde{a}}{\partial t}(t, r, \theta) = \alpha^{-1/4} \phi'(t - r) b(y, \theta). \quad (4.22)$$

and

$$\frac{\partial \tilde{a}}{\partial r}(t, r, \theta) = -\frac{1}{4} \alpha^{-5/4} \frac{\partial \alpha}{\partial r} \phi(t - r) b(y, \theta) - \alpha^{-1/4} \phi'(t - r) b(y, \theta). \quad (4.23)$$

Finally, (4.23) and (4.22) yield

$$\frac{\partial \tilde{a}}{\partial t}(t, r, \theta) + \frac{\partial \tilde{a}}{\partial r}(t, r, \theta) = -\frac{1}{4} \alpha^{-1} \tilde{a}(t, r, \theta) \frac{\partial \alpha}{\partial r}. \quad (4.24)$$

Now if we assume that  $\text{supp}(\phi) \subset (0, \varepsilon_0)$ ,  $\varepsilon_0 > 0$  small, then for any  $x = \exp_y(r\theta) \in \mathcal{M}$ , it is easy to see that  $\tilde{a}(t, r, \theta) = 0$  if  $t \leq 0$  and  $t \geq T_0$  for some  $T_0 > 0$  sufficiently large.

## 5 Stable determination of the electric potential

In this section, we complete the proof of Theorem 2. We are going to use the geometrical optics solutions constructed in the previous section; this will provide information on the geodesic ray transform of the difference of electric potentials.

### 5.1 Preliminary estimates

The main purpose of this section is to present a preliminary estimate, which relates the difference of the potentials to the Dirichlet-to-Neumann map. As before, we let  $q_1, q_2 \in \mathcal{Q}(M_0)$  such that  $q_1 = q_2$  on the boundary  $\partial\mathcal{M}$ . We set

$$q(x) = (q_1 - q_2)(x).$$

Recall that we have extended  $q_1, q_2$  as  $H^1(\mathcal{M}_2)$  in such a way that  $q = 0$  on  $\mathcal{M}_2 \setminus \mathcal{M}$ .

**Lemma 5.1** *There exists  $C > 0$  such that for any  $a_1, a_2 \in H^1(\mathbb{R}, H^2(\mathcal{M}))$  satisfying the transport equation (4.2) with (4.3), the following estimate holds true:*

$$\left| \int_0^T \int_{\mathcal{M}} q(x) a_1(2\lambda t, x) \overline{a_2}(2\lambda t, x) \, dv_g^n \, dt \right| \leq C \left( \lambda^{-2} + \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\| \right) \|a_1\|_* \|a_2\|_* \quad (5.1)$$

for any sufficiently large  $\lambda > 0$ .

**Proof .** First, if  $a_2$  satisfies (4.2), (4.3) and  $\lambda$  is large enough, Lemma 4.1 guarantees the existence of a geometrical optics solution  $u_2$

$$u_2(t, x) = a_2(2\lambda t, x) e^{i\lambda(\psi(x) - \lambda t)} + v_{2, \lambda}(t, x), \quad (5.2)$$

to the Schrödinger equation corresponding to the electric potential  $q_2$ ,

$$(i\partial_t + \Delta_g + q_2(x)) u(t, x) = 0 \quad \text{in } (0, T) \times \mathcal{M}, \quad u(0, \cdot) = 0 \quad \text{in } \mathcal{M}$$

where  $v_{2, \lambda}$  satisfies

$$\begin{aligned} \lambda \|v_{2, \lambda}(t, \cdot)\|_{L^2(\mathcal{M})} + \|\nabla v_{2, \lambda}(t, \cdot)\|_{L^2(\mathcal{M})} &\leq C \|a_2\|_* \\ v_{2, \lambda}(t, x) &= 0, \quad \forall (t, x) \in (0, T) \times \partial\mathcal{M}. \end{aligned} \quad (5.3)$$

Moreover

$$u_2 \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M})).$$

Let us denote by  $f_\lambda$  the function

$$f_\lambda(t, x) = a_2(2\lambda t, x)e^{i\lambda(\psi(x)-\lambda t)}, \quad t \in (0, T), \quad x \in \partial\mathcal{M}.$$

Let us consider  $v$  the solution of the following non-homogenous boundary value problem

$$\begin{cases} (i\partial_t + \Delta_g + q_1) v = 0, & (t, x) \in (0, T) \times \mathcal{M}, \\ v(0, x) = 0, & x \in \mathcal{M}, \\ v(t, x) = u_2(t, x) := f_\lambda(t, x), & (t, x) \in (0, T) \times \partial\mathcal{M}. \end{cases} \quad (5.4)$$

Denote  $w = v - u_2$ . Therefore,  $w$  solves the following homogenous boundary value problem for the Schrödinger equation

$$\begin{cases} (i\partial_t + \Delta_g + q_1(x)) w(t, x) = 0 & (t, x) \in (0, T) \times \mathcal{M}, \\ w(0, x) = 0, & x \in \mathcal{M}, \\ w(t, x) = 0, & (t, x) \in (0, T) \times \partial\mathcal{M}. \end{cases}$$

Using the fact that  $q(x)u_2 \in W^{1,1}(0, T; L^2(\mathcal{M}))$  with  $u(0, \cdot) \equiv 0$ , by Lemma 3.2, we deduce that

$$w \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})).$$

Therefore, we have constructed a special solution

$$u_1 \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M}))$$

to the backward Schrödinger equation

$$\begin{aligned} (i\partial_t + \Delta_g + q_1(x)) u_1(t, x) &= 0, & (t, x) \in (0, T) \times \mathcal{M}, \\ u_1(T, x) &= 0, & x \in \mathcal{M}, \end{aligned}$$

having the special form

$$u_1(t, x) = a_1(2\lambda t, x)e^{i\lambda(\psi(x)-\lambda t)} + v_{1,\lambda}(t, x), \quad (5.5)$$

which corresponds to the electric potential  $q_1$ , where  $v_{1,\lambda}$  satisfies

$$\lambda \|v_{1,\lambda}(t, \cdot)\|_{L^2(\mathcal{M})} + \|\nabla v_{1,\lambda}(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \|a_1\|_*. \quad (5.6)$$

Integrating by parts and using Green's formula (2.4), we find

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} (i\partial_t + \Delta_g + q_1) w \bar{u}_1 \, dv_g^n \, dt &= \int_0^T \int_{\mathcal{M}} q u_2 \bar{u}_1 \, dv_g^n \, dt \\ &= - \int_0^T \int_{\partial\mathcal{M}} \partial_\nu w \bar{u}_1 \, d\sigma_g^{n-1} \, dt. \end{aligned} \quad (5.7)$$

Taking (5.7), (5.4) into account, we deduce

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} q(x) u_2(t, x) \bar{u}_1(t, x) \, dv_g^n \, dt \\ = - \int_0^T \int_{\partial\mathcal{M}} (\Lambda_{g, q_1} - \Lambda_{g, q_2}) (f_\lambda)(t, x) \bar{g}_\lambda(t, x) \, d\sigma_g^{n-1} \, dt \end{aligned} \quad (5.8)$$

where  $g_\lambda$  is given by

$$g_\lambda(t, x) = a_1(2\lambda t, x) e^{i\lambda(\psi(x) - \lambda t)}, \quad (t, x) \in (0, T) \times \partial\mathcal{M}.$$

It follows from (5.8), (5.5) and (5.2) that

$$\begin{aligned} \int_0^T \int_{\mathcal{M}} q(x) (a_2 \bar{a}_1)(2\lambda t, x) \, dv_g^n \, dt &= - \int_0^T \int_{\partial\mathcal{M}} \bar{g}_\lambda (\Lambda_{g, q_1} - \Lambda_{g, q_2}) f_\lambda \, d\sigma_g^{n-1} \, dt \\ &\quad - \int_0^T \int_{\mathcal{M}} q e^{i\lambda(\psi - \lambda t)} a_2(2\lambda t, x) \bar{v}_{1, \lambda} \, dv_g^n \, dt \\ &\quad - \int_0^T \int_{\mathcal{M}} q v_{2, \lambda} e^{-i\lambda(\psi - \lambda t)} \bar{a}_1(2\lambda t, x) \, dv_g^n \, dt \\ &\quad - \int_0^T \int_{\mathcal{M}} q v_{2, \lambda} \bar{v}_{1, \lambda} \, dv_g^n \, dt. \end{aligned} \quad (5.9)$$

In view of (5.6) and (5.3), we have

$$\begin{aligned} \left| \int_0^T \int_{\mathcal{M}} q e^{i\lambda(\psi - \lambda t)} a_2(2\lambda t, x) \bar{v}_{1, \lambda} \, dv_g^n \, dt \right| \\ \leq C \int_0^T \|a_2(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \|v_{1, \lambda}(t, \cdot)\|_{L^2(\mathcal{M})} \, dt \\ \leq C \lambda^{-2} \|a_2\|_* \|a_1\|_*. \end{aligned} \quad (5.10)$$

Similarly, we deduce

$$\left| \int_0^T \int_{\mathcal{M}} q e^{-i\lambda(\psi - \lambda t)} \bar{a}_1(2\lambda t, x) v_{2, \lambda}(t, x) \, dv_g^n \, dt \right| \leq C \lambda^{-2} \|a_1\|_* \|a_2\|_*. \quad (5.11)$$

Moreover we have

$$\left| \int_0^T \int_{\mathcal{M}} q(x) v_{2,\lambda}(t, x) \bar{v}_{1,\lambda}(t, x) \, dv_g^n \, dt \right| \leq C \lambda^{-2} \|a_1\|_* \|a_2\|_*. \quad (5.12)$$

On the other hand, by the trace theorem, we find

$$\begin{aligned} & \left| \int_0^T \int_{\partial\mathcal{M}} (\Lambda_{g, q_1} - \Lambda_{g, q_2}) (f_\lambda) \bar{g}_\lambda \, d\sigma_g^{n-1} \, dt \right| \\ & \leq \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\| \|f_\lambda\|_{H^1((0,T) \times \partial\mathcal{M})} \|g_\lambda\|_{L^2((0,T) \times \partial\mathcal{M})} \\ & \leq C \frac{\lambda^{1/2}}{\lambda^{1/2}} \|a_1\|_* \|a_2\|_* \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|. \end{aligned} \quad (5.13)$$

The estimate (5.1) follows easily from (5.9), (5.10), (5.11), (5.12) and (5.13). This completes the proof of the Lemma.  $\square$

**Lemma 5.2** *Let  $M_0 > 0$ . There exists  $C > 0$  such that for any  $b \in H^2(\partial_+ S\mathcal{M}_1)$ , the following estimate*

$$\begin{aligned} & \left| \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(s, \theta) b(y, \theta) \mu(y, \theta) \, ds \, d\omega_y(\theta) \right| \\ & \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/2} \|b(y, \cdot)\|_{H^2(S_y^+ \mathcal{M}_1)}. \end{aligned} \quad (5.14)$$

holds for any  $y \in \partial\mathcal{M}_1$ .

We use the notation

$$S_y^+ \mathcal{M}_1 = \{\theta \in S_y \mathcal{M}_1 : \langle \nu, \theta \rangle_g < 0\}.$$

**Proof .** Following (4.21), we take two solutions to (4.2) and (4.3) of the form

$$\begin{aligned} \tilde{a}_1(t, r, \theta) &= \alpha^{-1/4} \phi(t - r) b(y, \theta), \\ \tilde{a}_2(t, r, \theta) &= \alpha^{-1/4} \phi(t - r) \mu(y, \theta). \end{aligned}$$

We recall that  $\mu(y, \theta) = \langle \nu(y), \theta \rangle$  is the density of the  $L^2$  space where the image of the geodesic ray transform lies. Now we change variable in (5.1),  $x = \exp_y(r\theta)$ ,

$r > 0$  and  $\theta \in S_y \mathcal{M}_1$ , we have

$$\begin{aligned}
& \int_0^T \int_{\mathcal{M}} q a_1(2\lambda t, x) a_2(2\lambda t, x) \, dv_g^n \, dt \\
&= \int_0^T \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(r, \theta) \tilde{a}_1(2\lambda t, r, \theta) \tilde{a}_2(2\lambda t, r, \theta) \alpha^{1/2} \, dr \, d\omega_y(\theta) \, dt \\
&= \int_0^T \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(r, \theta) \phi^2(2\lambda t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \\
&= \frac{1}{2\lambda} \int_0^{2\lambda T} \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt.
\end{aligned}$$

By virtue of Lemma 5.1, we conclude that

$$\begin{aligned}
& \left| \int_0^\infty \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \right| \\
& \leq C \left( \lambda^{-1} + \lambda \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\| \right) \|\phi\|_{H^3(\mathbb{R})}^2 \|b(y, \cdot)\|_{H^2(S_y^+ \mathcal{M}_1)}. \quad (5.15)
\end{aligned}$$

By the support properties of the function  $\phi$ , we get that the left-hand side term in the previous inequality reads

$$\begin{aligned}
& \int_0^\infty \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \\
&= \left( \int_{-\infty}^\infty \phi(t) \, dt \right) \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(r, \theta) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta).
\end{aligned}$$

Finally, minimizing in  $\lambda$  in the right hand-side of (5.15) we obtain

$$\begin{aligned}
& \left| \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{q}(s, \theta) b(y, \theta) \mu(y, \theta) \, ds \, d\omega_y(\theta) \right| \\
& \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/2} \|b(y, \cdot)\|_{H^2(S_y^+ \mathcal{M}_1)}.
\end{aligned}$$

This completes the proof of the lemma.  $\square$

## 5.2 End of the proof of the stability estimate

Let us now complete the proof of the stability estimate in Theorem 2. Using Lemma 5.2, for any  $y \in \partial\mathcal{M}_1$  and  $b \in H^2(\partial_+ S\mathcal{M})$  we have

$$\left| \int_{S_y \mathcal{M}_1} \mathcal{I}(q)(y, \theta) b(y, \theta) \mu(y, \theta) d\omega_y(\theta) \right| \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/2} \|b(y, \cdot)\|_{H^2(S_y^+ \mathcal{M}_1)}.$$

Integrating with respect to  $y \in \partial\mathcal{M}_1$  we obtain

$$\left| \int_{\partial_+ S\mathcal{M}_1} \mathcal{I}(q)(y, \theta) b(y, \theta) \langle \theta, \nu(y) \rangle d\sigma_g^{2n-2}(y, \theta) \right| \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/2} \|b\|_{H^2(\partial_+ S\mathcal{M}_1)}. \quad (5.16)$$

Now we choose

$$b(y, \theta) = \mathcal{I}(\mathcal{I}^* \mathcal{I}(q))(y, \theta).$$

Taking into account (2.14) and (2.10), we obtain

$$\|\mathcal{I}^* \mathcal{I}(q)\|_{L^2(\mathcal{M}_1)}^2 \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/2} \|q\|_{H^1(\mathcal{M})}.$$

By interpolation, it follows that

$$\begin{aligned} \|\mathcal{I}^* \mathcal{I}(q)\|_{H^1(\mathcal{M}_1)}^2 &\leq C \|\mathcal{I}^* \mathcal{I}(q)\|_{L^2(\mathcal{M}_1)} \|\mathcal{I}^* \mathcal{I}(q)\|_{H^2(\mathcal{M}_1)} \\ &\leq C \|\mathcal{I}^* \mathcal{I}(q)\|_{L^2(\mathcal{M}_1)} \|q\|_{H^1(\mathcal{M})} \\ &\leq C \|\mathcal{I}^* \mathcal{I}(q)\|_{L^2(\mathcal{M}_1)} \\ &\leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/4}. \end{aligned} \quad (5.17)$$

Using (2.13), we deduce that

$$\|q\|_{L^2(\mathcal{M})}^2 \leq C \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\|^{1/4}.$$

This completes the proof of Theorem 2.

## 6 Stable determination of the conformal factor

This section is devoted to the proof of the stability estimate for the conformal factor. We use the following notations; let  $c \in \mathcal{C}(M_0, k, \varepsilon)$ , we denote

$$\begin{aligned} \varrho_0(x) &= 1 - c(x), \quad \varrho_1(x) = c^{n/2}(x) - 1, \quad \varrho_2(x) = c^{n/2-1}(x) - 1, \\ \varrho(x) &= \varrho_2(x) - \varrho_1(x) = c^{n/2-1}(x)(1 - c(x)). \end{aligned} \quad (6.1)$$

Then the following holds

$$\begin{aligned} \|\varrho_j\|_{C^1(\mathcal{M})} &\leq C \|\varrho_0\|_{C^1(\mathcal{M})}, \quad j = 1, 2 \\ C^{-1} \|\varrho_0\|_{L^2(\mathcal{M})} &\leq \|\varrho\|_{L^2(\mathcal{M})} \leq C \|\varrho_0\|_{L^2(\mathcal{M})}. \end{aligned} \quad (6.2)$$

As in the case of potentials, we extend the manifold  $(\mathcal{M}, g)$  into a simple manifold  $\mathcal{M}_2 \ni \mathcal{M}$  so that  $\mathcal{M}_2 \ni \mathcal{M}_1 \ni \mathcal{M}$  with  $(\mathcal{M}_1, g)$  simple. We extend the conformal factor  $c$  by 1 outside the manifold  $\mathcal{M}$ ; its  $\mathcal{C}^k(\mathcal{M}_1)$  norms may also be bounded by  $M_0$ . The first step in our analysis is the following lemma.

**Lemma 6.1** *Let  $c \in \mathcal{C}^\infty(\mathcal{M})$  such that  $c = 1$  on the boundary  $\partial\mathcal{M}$ . Let  $u_1, u_2$  solve the following boundary problems in  $(0, T) \times \mathcal{M}$  with some  $T > 0$*

$$\begin{cases} (i\partial_t + \Delta_g)u_1 = 0, & \text{in } (0, T) \times \mathcal{M} \\ u_1(0, \cdot) = 0, & \text{in } \mathcal{M} \\ u_1 = f_1, & \text{on } (0, T) \times \partial\mathcal{M} \end{cases} \quad (6.3)$$

$$\begin{cases} (i\partial_t + \Delta_{cg})u_2 = 0, & \text{in } (0, T) \times \mathcal{M} \\ u_2(0, \cdot) = 0, & \text{in } \mathcal{M} \\ u_2 = f_2, & \text{on } (0, T) \times \partial\mathcal{M} \end{cases} \quad (6.4)$$

Then the following identity

$$\begin{aligned} \int_0^T \int_{\partial\mathcal{M}} (\Lambda_g - \Lambda_{cg}) f_1 \bar{f}_2 \, d\sigma_g^{n-1} \, dt &= i \int_0^T \int_{\mathcal{M}} \varrho_1(x) u_1 \partial_t \bar{u}_2 \, dv_g^n \, dt \\ &+ \int_0^T \int_{\mathcal{M}} \varrho_2(x) \langle \nabla_g u_1(t, x), \nabla_g \bar{u}_2(t, x) \rangle_g \, dv_g^n \, dt \end{aligned} \quad (6.5)$$

holds true for any  $f_j \in H^1((0, T) \times \partial\mathcal{M})$ ,  $j = 1, 2$ .

**Proof .** We multiply both hand sides of the first equation (6.3) by  $\bar{u}_2$ , integrate by parts in time and use Green's formula (2.4) to get

$$\begin{aligned} 0 &= \int_0^T \int_{\mathcal{M}} (i\partial_t u_1 + \Delta_g u_1) \bar{u}_2 \, dv_g^n \, dt \\ &= -i \int_0^T \int_{\mathcal{M}} u_1 \partial_t \bar{u}_2 \, dv_g^n \, dt + i \int_0^T \int_{\mathcal{M}} \varrho_1 u_1 \partial_t \bar{u}_2 \, dv_g^n \, dt \end{aligned}$$



$$\begin{aligned}
& + \int_0^T \int_{\partial\mathcal{M}} \partial_\nu u_1 \overline{f_2} \, d\sigma_g^{n-1} dt - \int_0^T \int_{\mathcal{M}} \sum_{j,k=1}^n c g^{jk} \left( \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u_2}}{\partial x_k} \right) \, dv_{cg}^n dt \\
& + \int_0^T \int_{\mathcal{M}} \varrho_2 \left( \sum_{j,k=1}^n c g^{jk} \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u_2}}{\partial x_k} \right) \, dv_g^n dt
\end{aligned}$$

and after using a second time Green's formula, we end up with

$$\begin{aligned}
0 & = i \int_0^T \int_{\mathcal{M}} \varrho_1 u_1 \partial_t \overline{u_2} \, dv_g^n dt + \int_0^T \int_{\mathcal{M}} \varrho_2 \left( \sum_{j,k=1}^n c g^{jk} \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u_2}}{\partial x_k} \right) \, dv_g^n dt \\
& + \int_0^T \int_{\mathcal{M}} u_1 (-i \partial_t \overline{u_2} + \Delta_{cg} \overline{u_2}) \, dv_{cg}^n dt - \int_0^T \int_{\partial\mathcal{M}} \partial_\nu \overline{u_2} f_1 \, d\sigma_{cg}^{n-1} dt \\
& + \int_0^T \int_{\partial\mathcal{M}} \partial_\nu u_1 \overline{f_2} \, d\sigma_g^{n-1} dt.
\end{aligned}$$

Taking into account the fact that  $c = 1$  on  $\partial\mathcal{M}$ , the fact that  $(-i \partial_t \overline{u_2} + \Delta_{cg} \overline{u_2}) = 0$  in  $(0, T) \times \mathcal{M}$ , and the fact that the Dirichlet-to-Neumann map is selfadjoint, it follows that

$$\begin{aligned}
\int_0^T \int_{\partial\mathcal{M}} (\Lambda_g - \Lambda_{cg}) f_1 \overline{f_2} \, d\sigma_g^{n-1} dt & = i \int_0^T \int_{\mathcal{M}} \varrho_1(x) u_1 \partial_t \overline{u_2} \, dv_g^n dt \\
& + \int_0^T \int_{\mathcal{M}} \varrho_2(x) \left( \sum_{j,k=1}^n g^{jk} \frac{\partial u_1}{\partial x_j} \frac{\partial \overline{u_2}}{\partial x_k} \right) \, dv_g^n dt \quad (6.6)
\end{aligned}$$

This completes the proof of the Lemma.  $\square$

## 6.1 Modified geometrical optics solutions

Let  $\psi_1, \psi_2$  be two phase functions solving the eikonal equation with respect to the metrics  $g$  and  $cg$ .

$$\begin{aligned}
|\nabla_g \psi_1|_g^2 & = \sum_{j,k=1}^n g^{jk} \frac{\partial \psi_1}{\partial x_j} \frac{\partial \psi_1}{\partial x_k} = 1, \\
|\nabla_{cg} \psi_2|_{cg}^2 & = \sum_{j,k=1}^n c g^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial \psi_2}{\partial x_k} = 1,
\end{aligned} \quad \text{on } \mathcal{M}. \quad (6.7)$$

Let  $a_2$  solve the transport equation in  $\mathbb{R} \times \mathcal{M}$  with respect the metric  $g$  (as given in section 4)

$$\frac{\partial a_2}{\partial t} + \sum_{j,k=1}^n g^{jk} \frac{\partial \psi_1}{\partial x_j} \frac{\partial a_2}{\partial x_k} + \frac{a_2}{2} \Delta_g \psi_1 = 0. \quad (6.8)$$

Let  $a_3$  solve the following transport equation in  $\mathbb{R} \times \mathcal{M}$  with respect to the metric  $cg$

$$\begin{aligned} \frac{\partial a_3}{\partial t} + \sum_{j,k=1}^n (cg)^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial a_3}{\partial x_k} + \frac{a_3}{2} \Delta_{cg} \psi_2 &= -\frac{1}{2i} a_2(t, x) (1 - c^{-1}) e^{i\lambda(\psi_1 - \psi_2)} \\ &\equiv a_2(t, x) \varphi_0(x, \lambda) \end{aligned} \quad (6.9)$$

which satisfies the bound

$$\|a_3\|_* \leq C\lambda \|\varrho_0\|_{C^1(\mathcal{M})} \|a_2\|_*. \quad (6.10)$$

Let us now explain how to construct a solution  $a_3$  satisfying (6.9) and (6.10). To solve the transport equation (6.9) and (6.10) it is enough to take, in the geodesic polar coordinates  $(r, \theta)$  (with respect to the metric  $cg$ )

$$\tilde{a}_3(t, r, \theta; \lambda) = \alpha_{cg}^{-1/4}(r, \theta) \int_0^r \alpha_{cg}^{1/4}(s, \theta) \tilde{a}_2(s - r + t, s, \theta) \tilde{\varphi}_0(s, \theta, \lambda) ds, \quad (6.11)$$

where  $\alpha_{cg}(r, \theta)$  denotes the square of the volume element in geodesic polar coordinates with respect to the metric  $cg$ . Using that  $\|\varphi_0(\cdot, \lambda)\|_{C^1(\mathcal{M})} \leq C\lambda \|\varrho_0\|_{C^1(\mathcal{M})}$  and (6.11) we obtain (6.10).

**Lemma 6.2** *Let  $c \in \mathcal{C}(M_0, k, \varepsilon)$  be such that  $c = 1$  near the boundary  $\partial\mathcal{M}$ . Then the equation*

$$(i\partial_t + \Delta_{cg}) u = 0, \quad \text{in } (0, T) \times \mathcal{M}, \quad u(0, x) = 0 \quad (6.12)$$

*has a solution of the form*

$$u_2(t, x) = \frac{1}{\lambda} a_2(2\lambda t, x) e^{i\lambda(\psi_1(x) - \lambda t)} + a_3(2\lambda t, x; \lambda) e^{i\lambda(\psi_2(x) - \lambda t)} + v_{2,\lambda}(t, x) \quad (6.13)$$

*which satisfies*

$$\begin{aligned} \lambda \|v_{2,\lambda}(t, \cdot)\|_{L^2(\mathcal{M})} + \|\nabla v_{2,\lambda}(t, \cdot)\|_{L^2(\mathcal{M})} + \lambda^{-1} \|\partial_t v_{2,\lambda}(t, \cdot)\|_{L^2(\mathcal{M})} \\ \leq C \left( \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})} + \lambda^{-1} \right) \|a_2\|_* \end{aligned} \quad (6.14)$$

*where the constant  $C$  depends only on  $T$  and  $\mathcal{M}$  (that is  $C$  does not depend on  $a$ ,  $\lambda$  and  $\varepsilon$ ).*

**Proof .** We set

$$k(t, x) = -(i\partial_t + \Delta_{cg}) \left( \frac{1}{\lambda} a(2\lambda t, x) e^{i\lambda(\psi_1 - \lambda t)} - a_3(2\lambda t, x, \lambda) e^{i\lambda(\psi_2 - \lambda t)} \right).$$

To prove our Lemma it is enough to show that if  $v$  solves

$$(i\partial_t + \Delta_{cg}) v = k \quad (6.15)$$

with initial and boundary conditions

$$v(0, x) = 0, \quad \text{in } \mathcal{M}, \quad \text{and} \quad v(t, x) = 0 \quad \text{on } (0, T) \times \partial\mathcal{M} \quad (6.16)$$

then the estimate (6.14) holds. But we have

$$\begin{aligned} -k(t, x) &= \frac{1}{\lambda} e^{i\lambda(\psi_1 - \lambda t)} \Delta_{cg} a_2(2\lambda t, x) \\ &\quad + 2ie^{i\lambda(\psi_1 - \lambda t)} \left( \partial_t a_2 + \sum_{j,k=1}^n cg^{jk} \frac{\partial \psi_1}{\partial x_j} \frac{\partial a_2}{\partial x_k} + \frac{a_2}{2} \Delta_{cg} \psi_1 \right) (2\lambda t, x) \\ &\quad + \lambda e^{i\lambda(\psi_1 - \lambda t)} a_2(2\lambda t, x) \left( 1 - c^{-1} \sum_{j,k=1}^n g^{jk} \frac{\partial \psi_1}{\partial x_j} \frac{\partial \psi_1}{\partial x_k} \right) \\ &\quad + e^{i\lambda(\psi_2 - \lambda t)} (\Delta_{cg}) (a_3(2\lambda t, x)) \\ &\quad + 2i\lambda e^{i\lambda(\psi_2 - \lambda t)} \left( \partial_t a_3 + \sum_{j,k=1}^n cg^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial a_3}{\partial x_k} + \frac{a_3}{2} \Delta_{cg} \psi_2 \right) (2\lambda t, x) \\ &\quad + \lambda^2 e^{i\lambda(\psi_2 - \lambda t)} a_3(2\lambda t, x) \left( 1 - \sum_{j,k=1}^n (cg)^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial \psi_2}{\partial x_k} \right). \end{aligned} \quad (6.17)$$

Taking into account (6.7) and (6.8), the right-hand side of (6.17) becomes

$$\begin{aligned} -k(t, x) &= \frac{1}{\lambda} e^{i\lambda(\psi_1 - \lambda t)} \Delta_{cg} a_2(2\lambda t, x) \\ &\quad + 2ie^{i\lambda(\psi_1 - \lambda t)} \left( (c^{-1} - 1) \langle \nabla_g \psi_1, \nabla_g a_2(2\lambda t, x) \rangle_g \right. \\ &\quad \quad \left. + \frac{1}{2} a_2(2\lambda t, x) (\Delta_{cg} \psi_1 - \Delta_g \psi_1) \right) \\ &\quad + 2i\lambda e^{i\lambda(\psi_2 - \lambda t)} \left( \partial_t a_3 + \sum_{j,k=1}^n cg^{jk} \frac{\partial \psi_2}{\partial x_j} \frac{\partial a_3}{\partial x_k} + \frac{a_3}{2} \Delta_{cg} \psi_2 \right. \\ &\quad \quad \left. + \frac{a_2}{2i} e^{i\lambda(\psi_1 - \psi_2)} (1 - c^{-1}) \right) (2\lambda t, x) \end{aligned} \quad (6.18)$$

$$+ e^{i\lambda(\psi_2 - \lambda t)} \Delta_{cg} a_3(2\lambda t, x).$$

By (6.9) we get

$$\begin{aligned} -k(t, x) &= \frac{1}{\lambda} e^{i\lambda(\psi_1 - \lambda t)} \Delta_{cg} a_2(2\lambda t, x) \\ &\quad + 2ie^{i\lambda(\psi_1 - \lambda t)} \left( (c^{-1} - 1) \langle \nabla_g \psi_1, \nabla_g a_2(2\lambda t, x) \rangle_g \right. \\ &\quad \left. + \frac{1}{2} a_2(2\lambda t, x) (\Delta_{cg} \psi_1 - \Delta_g \psi_1) \right) \\ &\quad + e^{i\lambda(\psi_2 - \lambda t)} \Delta_{cg} a_3(2\lambda t, x) \\ &\equiv \frac{1}{\lambda} e^{i\lambda(\psi_1 - \lambda t)} k_0(2\lambda t, x) + e^{i\lambda(\psi_1 - \lambda t)} k_1(2\lambda t, x) + e^{i\lambda(\psi_2 - \lambda t)} k_2(2\lambda t, x). \end{aligned}$$

Since  $k_j \in H_0^1(0, T; L^2(\mathcal{M}))$ , by Lemma 3.2, we deduce that

$$v_\lambda \in \mathcal{C}^1(0, T; L^2(\mathcal{M})) \cap \mathcal{C}(0, T; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})) \quad (6.19)$$

and

$$\begin{aligned} &\|v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} \\ &\leq C \int_0^T \left( \frac{1}{\lambda} \|k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \|k_1(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \|k_2(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right) dt \\ &\leq \frac{C}{\lambda} \int_{\mathbb{R}} \left( \frac{1}{\lambda} \|k_0(s, \cdot)\|_{L^2(\mathcal{M})} + \|k_1(s, \cdot)\|_{L^2(\mathcal{M})} + \|k_2(s, \cdot)\|_{L^2(\mathcal{M})} \right) ds \\ &\leq \frac{C}{\lambda} \left( \frac{1}{\lambda} \|a_2\|_* + \|\varrho_0\|_{\mathcal{C}^1(M)} \|a_2\|_* + \lambda^2 \|\varrho_0\|_{\mathcal{C}^1(M)} \|a_2\|_* \right) \\ &\leq C \left( \lambda \|\varrho_0\|_{\mathcal{C}^1(M)} + \frac{1}{\lambda^2} \right) \|a_2\|_*. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \|\nabla v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} &\leq C\eta \left\{ \int_0^T \left( \lambda \|k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \|\partial_t k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right) dt \right. \\ &\quad + \int_0^T \left( \lambda^2 \|k_1(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \lambda \|\partial_t k_1(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right) dt \\ &\quad + \int_0^T \left( \lambda^2 \|k_2(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \lambda \|\partial_t k_2(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right) dt \Big\} \\ &\quad + \frac{C}{\eta} \int_0^T \left( \frac{1}{\lambda} \|k_0(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} + \|k_1(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right. \\ &\quad \left. + \|k_2(2\lambda t, \cdot)\|_{L^2(\mathcal{M})} \right) dt. \end{aligned}$$

Hence, we obtain the following estimate

$$\begin{aligned}
& \|\nabla v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} \\
& \leq C\eta \left\{ \int_{\mathbb{R}} \left( \|k_0(s, \cdot)\|_{L^2(\mathcal{M})} + \frac{1}{\lambda} \|\partial_t k_0(s, \cdot)\|_{L^2(\mathcal{M})} \right) dt \right. \\
& \quad + \int_{\mathbb{R}} \left( \lambda \|k_1(s, \cdot)\|_{L^2(\mathcal{M})} + \|\partial_t k_1(s, \cdot)\|_{L^2(\mathcal{M})} \right) dt \\
& \quad + \int_{\mathbb{R}} \left( \lambda \|k_2(s, \cdot)\|_{L^2(\mathcal{M})} + \|\partial_t k_2(s, \cdot)\|_{L^2(\mathcal{M})} \right) dt \Big\} \\
& \quad + \frac{C}{\eta} \int_{\mathbb{R}} \left( \frac{1}{\lambda^2} \|k_0(s, \cdot)\|_{L^2(\mathcal{M})} + \frac{1}{\lambda} \|k_1(s, \cdot)\|_{L^2(\mathcal{M})} \right. \\
& \quad \quad \quad \left. + \frac{1}{\lambda} \|k_2(s, \cdot)\|_{L^2(\mathcal{M})} \right) dt \\
& \leq C\eta \left( 1 + \lambda^3 \|\varrho_0\|_{C^1(M)} \right) \|a_2\|_* + \frac{C}{\eta} \left( \frac{1}{\lambda^2} + \lambda \|\varrho_0\|_{C^1(M)} \right) \|a_2\|_*.
\end{aligned} \tag{6.20}$$

Now choosing  $\eta = \lambda^{-1}$ , we obtain

$$\|\nabla v_\lambda(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \left( \frac{1}{\lambda} + \lambda^2 \|\varrho_0\|_{C^1(M)} \right) \|a_2\|_*. \tag{6.21}$$

Finally, if we study the equation satisfied by  $\partial_t v$ , we also find

$$\|\partial_t v(t, \cdot)\|_{L^2(\mathcal{M})} \leq C \left( 1 + \lambda^3 \|\varrho_0\|_{C^1(M)} \right) \|a_2\|_*. \tag{6.22}$$

This ends the proof of Lemma 6.2.  $\square$

**Lemma 6.3** *There exists  $C > 0$  such that for any  $a_1, a_2 \in H^1(\mathbb{R}, H^2(\mathcal{M}))$  satisfying the transport equation (6.8) with (4.3), the following estimate holds true*

$$\begin{aligned}
& \lambda \left| \int_0^T \int_{\mathcal{M}} \varrho(x) (a_1 \bar{a}_2) (2\lambda t, x) dv_g^n dt \right| \\
& \leq C \left\{ \|\varrho_0\|_{C^1(\mathcal{M})} \left( \lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})} \right) + \lambda \|\Lambda_g - \Lambda_{cg}\| \right\} \|a_1\|_* \|a_2\|_*
\end{aligned} \tag{6.23}$$

for any sufficiently large  $\lambda$ .

**Proof .** Following Lemma 6.2 let  $u_2$  be a solution to the problem  $(i\partial_t + \Delta_{cg})u = 0$  of the form

$$\bar{u}_2(t, x) = \frac{1}{\lambda} \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} + \bar{a}_3(2\lambda t, x; \lambda) e^{-i\lambda(\psi_2 - \lambda t)} + \bar{v}_{2,\lambda}(t, x)$$

where  $v_{2,\lambda}$  satisfies (6.14) and  $a_3$  satisfies (6.10). Thanks to Lemma 4.1 let  $u_1$  be a solution of  $(i\partial_t + \Delta_g)u = 0$  of the form

$$u_1(t, x) = a_1(2\lambda t, x) e^{i\lambda(\psi_1 - \lambda t)} + v_{1,\lambda}(t, x).$$

where  $v_{1,\lambda}$  satisfies (4.4). Then we have

$$\begin{aligned} \partial_t \bar{u}_2(t, x) &= 2\partial_t \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} + i\lambda \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} \\ &\quad + 2\lambda \partial_t \bar{a}_3(2\lambda t, x; \lambda) e^{-i\lambda(\psi_2 - \lambda t)} + i\lambda^2 \bar{a}_3(2\lambda t, x, \lambda) e^{-i\lambda(\psi_2 - \lambda t)} \\ &\quad + \partial_t \bar{v}_{2,\lambda}(t, x). \end{aligned} \quad (6.24)$$

Let us compute the first term in the right hand side of (6.5). We have

$$\int_0^T \int_{\mathcal{M}} \varrho_1 u_1 \partial_t \bar{u}_2 \, dv_g^n \, dt = i\lambda \int_0^T \int_{\mathcal{M}} \varrho_1 (a_1 \bar{a}_2)(2\lambda t, x) \, dv_g^n \, dt + \mathcal{J}_1(\lambda) + \mathcal{J}_2(\lambda) \quad (6.25)$$

with

$$\begin{aligned} \mathcal{J}_1(\lambda) &= 2 \int_0^T \int_{\mathcal{M}} \varrho_1 (a_1 \partial_t \bar{a}_2)(2\lambda t, x) \, dv_g^n \, dt \\ &\quad + 2 \int_0^T \int_{\mathcal{M}} \varrho_1 v_{1,\lambda} \partial_t \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} \, dv_g^n \, dt \\ &\quad + i\lambda \int_0^T \int_{\mathcal{M}} \varrho_1 v_{1,\lambda} \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} \, dv_g^n \, dt \end{aligned}$$

and with

$$\begin{aligned} \mathcal{J}_2(\lambda) &= +2\lambda \int_0^T \int_{\mathcal{M}} \varrho_1 (a_1 \partial_t \bar{a}_3)(2\lambda t, x) e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt \\ &\quad + i\lambda^2 \int_0^T \int_{\mathcal{M}} \varrho_1 (a_1 \bar{a}_3)(2\lambda t, x) e^{i\lambda(\psi_1 - \psi_2)} \, dv_g^n \, dt \\ &\quad + \int_0^T \int_{\mathcal{M}} \varrho_1 a_1(2\lambda t, x) \partial_t \bar{v}_{2,\lambda}(t, x) \, dv_g^n \, dt \\ &\quad + 2\lambda \int_0^T \int_{\mathcal{M}} \varrho_1 v_{1,\lambda} \partial_t \bar{a}_3(2\lambda t, x) e^{-i\lambda(\psi_2 - \lambda t)} \, dv_g^n \, dt \end{aligned}$$

$$\begin{aligned}
& + i\lambda^2 \int_0^T \int_{\mathcal{M}} \varrho_1 v_{1,\lambda} \bar{a}_3(2\lambda t, x) e^{-i\lambda(\psi_2 - \lambda t)} \, dv_g^n \, dt \\
& + \int_0^T \int_{\mathcal{M}} \varrho_1 v_{1,\lambda} \partial_t \bar{v}_{2,\lambda}(t, x) \, dv_g^n \, dt.
\end{aligned}$$

From (6.10), (6.14) and (4.4) we have the estimates

$$\begin{aligned}
|\mathcal{J}_1(\lambda)| & \leq C \|\varrho_0\|_{C^1(\mathcal{M})} \lambda^{-1} \|a_2\|_* \|a_1\|_* \\
|\mathcal{J}_2(\lambda)| & \leq C \|\varrho_0\|_{C^1(\mathcal{M})} \left( \lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})} \right) \|a_2\|_* \|a_1\|_*. \quad (6.26)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\nabla_g u_1 & = (\nabla_g a_1)(2\lambda t, x) e^{i\lambda(\psi_1 - \lambda t)} + i\lambda (\nabla_g \psi_1) a_1(2\lambda t, x) e^{i\lambda(\psi_1 - \lambda t)} + \nabla_g v_{1,\lambda} \\
\nabla_g \bar{u}_2 & = \frac{1}{\lambda} (\nabla_g \bar{a}_2)(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} - i\bar{a}_2(2\lambda t, x) \nabla_g \psi_1 e^{-i\lambda(\psi_1 - \lambda t)} \\
& \quad - i\lambda \bar{a}_3(2\lambda t, x) \nabla_g \psi_2 e^{-i\lambda(\psi_2 - \lambda t)} + \nabla_g \bar{a}_3(2\lambda t, x) e^{-i\lambda(\psi_2 - \lambda t)} + \nabla_g \bar{v}_{2,\lambda}
\end{aligned}$$

and the second term in the right-hand side of (6.5) becomes

$$\begin{aligned}
& \int_0^T \int_{\mathcal{M}} \varrho_2(x) \langle \nabla_g u_1(t, x), \nabla_g \bar{u}_2(t, x) \rangle_g \, dv_g^n \, dt \\
& = \lambda \int_0^T \int_{\mathcal{M}} \varrho_2(x) (a_1 \bar{a}_2)(2\lambda t, x) \, dv_g^n \, dt + \mathcal{J}_3(\lambda) + \mathcal{J}_4(\lambda) \quad (6.27)
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{J}_3(\lambda) & = \frac{1}{\lambda} \int_0^T \int_{\mathcal{M}} \varrho_2(x) \langle \nabla_g a_1(2\lambda t, x), \nabla_g \bar{a}_2(2\lambda t, x) \rangle_g \, dv_g^n \, dt \\
& \quad - i \int_0^T \int_{\mathcal{M}} \varrho_2(x) \bar{a}_2(2\lambda t, x) \langle \nabla_g a_1(2\lambda t, x), \nabla_g \psi_1(x) \rangle_g \, dv_g^n \, dt \\
& \quad + i \int_0^T \int_{\mathcal{M}} \varrho_2(x) a_1(2\lambda t, x) \langle \nabla_g \bar{a}_2(2\lambda t, x), \nabla_g \psi_1(x) \rangle_g \, dv_g^n \, dt \\
& \quad + \frac{1}{\lambda} \int_0^T \int_{\mathcal{M}} \varrho_2(x) e^{-i\lambda(\psi_1 - \lambda t)} \langle \nabla_g \bar{a}_2(2\lambda t, x), \nabla_g v_{1,\lambda}(t, x) \rangle_g \, dv_g^n \, dt \\
& \quad - i \int_0^T \int_{\mathcal{M}} \varrho_2(x) \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} \langle \nabla_g v_{1,\lambda}(t, x), \nabla_g \psi_1(x) \rangle_g \, dv_g^n \, dt
\end{aligned}$$

and with

$$\mathcal{J}_4(\lambda) = -i\lambda \int_0^T \int_{\mathcal{M}} \varrho_2(x) \bar{a}_3(2\lambda t, x) e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g a_1(2\lambda t, x), \nabla_g \psi_2(x) \rangle_g \, dv_g^n \, dt$$

$$\begin{aligned}
& + \int_0^T \int_{\mathcal{M}} \varrho_2(x) e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g a_1(2\lambda t, x), \nabla_g \bar{a}_3(2\lambda t, x) \rangle_g \, dv_g^n \, dt \\
& + \int_0^T \int_{\mathcal{M}} \varrho_2(x) e^{i\lambda(\psi_1 - \lambda t)} \langle \nabla_g a_1(2\lambda t, x), \nabla_g \bar{v}_{2,\lambda}(t, x) \rangle_g \, dv_g^n \, dt \\
& + \lambda^2 \int_0^T \int_{\mathcal{M}} \varrho_2(x) (a_1 \bar{a}_3)(2\lambda t, x) e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g \psi_1(x), \nabla_g \psi_2(x) \rangle_g \, dv_g^n \, dt \\
& + i\lambda \int_0^T \int_{\mathcal{M}} \varrho_2(x) a_1(2\lambda t, x) e^{i\lambda(\psi_1 - \psi_2)} \langle \nabla_g \bar{a}_3(2\lambda t, x), \nabla_g \psi_1(x) \rangle_g \, dv_g^n \, dt \\
& + i\lambda \int_0^T \int_{\mathcal{M}} \varrho_2(x) a_1(2\lambda t, x) e^{i\lambda(\psi_1 - \lambda t)} \langle \nabla_g \psi_1(x), \nabla_g \bar{v}_2(t, x) \rangle_g \, dv_g^n \, dt \\
& - i \int_0^T \int_{\mathcal{M}} \varrho_2 \bar{a}_2(2\lambda t, x) e^{-i\lambda(\psi_1 - \lambda t)} \langle \nabla_g v_{1,\lambda}(t, x), \nabla_g \psi_1(x) \rangle_g \, dv_g^n \, dt \\
& - i\lambda \int_0^T \int_{\mathcal{M}} \varrho_2 \bar{a}_3(2\lambda t, x) e^{-i\lambda(\psi_2 - \lambda t)} \langle \nabla_g v_{1,\lambda}(t, x), \nabla_g \psi_2 \rangle_g \, dv_g^n \, dt \\
& + \int_0^T \int_{\mathcal{M}} \varrho_2 e^{-i\lambda(\psi_2 - \lambda t)} \langle \nabla_g v_{1,\lambda}, \nabla_g \bar{a}_3(2\lambda t, x) \rangle_g \, dv_g^n \, dt \\
& + \int_0^T \int_{\mathcal{M}} \varrho_2 \langle \nabla_g v_{1,\lambda}, \nabla_g \bar{v}_{2,\lambda} \rangle_g \, dv_g^n \, dt.
\end{aligned}$$

From (6.10), (6.14) and (4.4), we have

$$\begin{aligned}
|\mathcal{J}_3(\lambda)| & \leq \|\varrho_0\|_{C^1(\mathcal{M})} \lambda^{-1} \|a_2\|_* \|a_1\|_* \\
|\mathcal{J}_4(\lambda)| & \leq \|\varrho_0\|_{C^1(\mathcal{M})} \left( \lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})} \right) \|a_2\|_* \|a_1\|_*.
\end{aligned} \tag{6.28}$$

Taking into account (6.5), (6.25) and (6.27), we deduce that

$$\begin{aligned}
\int_0^T \int_{\partial\mathcal{M}} (\Lambda_g - \Lambda_{cg}) f_1 \bar{f}_2 \, d\sigma_g^{n-1} \, dt & = \lambda \int_0^T \int_{\mathcal{M}} \varrho(x) (a_1 \bar{a}_2)(2\lambda t, x) \, dv_g^n \, dt \\
& + \mathcal{J}_1(\lambda) + \mathcal{J}_2(\lambda) + \mathcal{J}_3(\lambda) + \mathcal{J}_4(\lambda). \tag{6.29}
\end{aligned}$$

In view of (6.26) and (6.28), we obtain

$$\begin{aligned}
& \lambda \left| \int_0^T \int_{\mathcal{M}} \varrho(x) (a_1 \bar{a}_2)(2\lambda t, x) \, dv_g^n \, dt \right| \\
& \leq C \left\{ \|\varrho_0\|_{C^1(\mathcal{M})} \left( \lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})} \right) + \lambda \|\Lambda_g - \Lambda_{cg}\| \right\} \|a_1\|_* \|a_2\|_*.
\end{aligned}$$

This completes the proof.  $\square$



## 6.2 Stability estimate for the geodesic ray transform

**Lemma 6.4** *There exists  $C > 0$  such that for any  $b \in H^2(\partial_+ S\mathcal{M}_1)$  the following estimate*

$$\begin{aligned} & \left| \int_{\partial_+ S\mathcal{M}_1} \mathcal{I}(\varrho)(y, \theta) b(y, \theta) \langle \theta, \nu(y) \rangle \, d\sigma_g^{2n-2}(y, \theta) \right| \\ & \leq C \left( (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}) \|\varrho_0\|_{C^1(\mathcal{M})} + \lambda \|\Lambda_g - \Lambda_{cg}\| \right) \|b\|_{H^2(\partial_+ S\mathcal{M}_1)} \end{aligned} \quad (6.30)$$

holds for any  $y \in \partial\mathcal{M}_1$ .

**Proof .** Following (4.21), we take two solutions of the form

$$\begin{aligned} \tilde{a}_1(t, r, \theta) &= \alpha^{-1/4} \phi(t - r) b(y, \theta), \\ \tilde{a}_2(t, r, \theta) &= \alpha^{-1/4} \phi(t - r) \mu(y, \theta). \end{aligned}$$

Now we change variable in (6.23),  $x = \exp_y(r\theta)$ ,  $r > 0$  and  $\theta \in S_y\mathcal{M}_1$ . Then

$$\begin{aligned} & \int_0^T \int_{\mathcal{M}} \varrho a_1(2\lambda t, x) a_2(2\lambda t, x) \, dv_g^n \, dt \\ &= \int_0^T \int_{S_y\mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{\varrho}(r, \theta) \tilde{a}_1(2\lambda t, r, \theta) \tilde{a}_2(2\lambda t, r, \theta) \alpha^{1/2} \, dr \, d\omega_y(\theta) \, dt \\ &= \int_0^T \int_{S_y\mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{\varrho}(r, \theta) \phi^2(2\lambda t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \\ &= \frac{1}{2\lambda} \int_0^{2\lambda T} \int_{S_y\mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{\varrho}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt. \end{aligned}$$

We conclude that

$$\begin{aligned} & \left| \int_0^T \int_{S_y\mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{\varrho}(r, \theta) \phi^2(t - r) b(y, \theta) \mu(y, \theta) \, dr \, d\omega_y(\theta) \, dt \right| \\ & \leq C \left( (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}) \|\varrho_0\|_{C^1(\mathcal{M})} \right. \\ & \quad \left. + \lambda \|\Lambda_{g, q_1} - \Lambda_{g, q_2}\| \right) \|b(y, \cdot)\|_{H^2(S_y^+ \mathcal{M}_1)} \end{aligned} \quad (6.31)$$

where  $S_y^+ \mathcal{M}_1 = \{\theta \in S_y \mathcal{M} : \langle \theta, \nu \rangle_g < 0\}$ . Given the support properties of the function  $\phi$ , the left-hand side of the inequality reads in fact

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{\varrho}(r, \theta) \phi^2(t-r) b(y, \theta) \mu(y, \theta) dr d\omega_y(\theta) dt \\ &= \left( \int_{-\infty}^{\infty} \phi^2(t) dt \right) \times \int_{S_y \mathcal{M}_1} \int_0^{\tau_+(y, \theta)} \tilde{\varrho}(r, \theta) b(y, \theta) \mu(y, \theta) dr d\omega_y(\theta). \end{aligned}$$

Integrating with respect to  $y \in \partial \mathcal{M}_1$  in (6.31) we obtain

$$\begin{aligned} & \left| \int_{\partial_+ S \mathcal{M}_1} \mathcal{I}(\varrho)(y, \theta) b(y, \theta) \langle \theta, \nu(y) \rangle d\sigma_g^{2n-2}(y, \theta) \right| \\ & \leq \left( (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}) \|\varrho_0\|_{C^1(\mathcal{M})} + \lambda \|\Lambda_g - \Lambda_{cg}\| \right) \|b\|_{H^2(\partial_+ S \mathcal{M}_1)}. \end{aligned}$$

This completes the proof of the lemma.  $\square$

### 6.3 End of the proof of Theorem 3

Let us now prove Theorem 3. We choose

$$b(y, \theta) = \mathcal{I}(\mathcal{I}^* \mathcal{I}(q))(y, \theta)$$

and obtain using Lemma 6.4 and (2.10)

$$\begin{aligned} \|\mathcal{I}^* \mathcal{I}(\varrho)\|_{L^2(\mathcal{M}_1)}^2 & \leq C \left( (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}) \|\varrho_0\|_{C^1(\mathcal{M})} \right. \\ & \quad \left. + \lambda \|\Lambda_g - \Lambda_{cg}\| \right) \|\mathcal{I}^* \mathcal{I}(\varrho)\|_{H^2(\mathcal{M}_1)}. \end{aligned}$$

By interpolation we have

$$\begin{aligned} \|\mathcal{I}^* \mathcal{I}(\varrho)\|_{H^1(\mathcal{M}_1)}^2 & \leq C \|\mathcal{I}^* \mathcal{I}(\varrho)\|_{L^2(\mathcal{M}_1)} \|\mathcal{I}^* \mathcal{I}(\varrho)\|_{H^2(\mathcal{M}_1)} \\ & \leq C \left( (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}) \|\varrho_0\|_{C^1(\mathcal{M})} \right. \\ & \quad \left. + \lambda \|\Lambda_g - \Lambda_{cg}\| \right)^{1/2} \|\mathcal{I}^* \mathcal{I}(\varrho)\|_{H^2(\mathcal{M})}^{3/2}. \end{aligned}$$

We use (2.13) and (2.14) to deduce

$$\begin{aligned} \|\varrho\|_{L^2(\mathcal{M})}^2 & \lesssim \left( (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}) \|\varrho_0\|_{C^1(\mathcal{M})} + \lambda \|\Lambda_g - \Lambda_{cg}\| \right)^{\frac{1}{2}} \|\varrho\|_{H^1(\mathcal{M})}^{\frac{3}{2}} \\ & \lesssim (\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})})^{\frac{1}{2}} \|\varrho_0\|_{C^1(\mathcal{M})}^2 + \lambda^{\frac{1}{2}} \|\varrho_0\|_{C^1(\mathcal{M})}^{\frac{3}{2}} \|\Lambda_g - \Lambda_{cg}\|^{\frac{1}{2}}. \end{aligned}$$

Minimizing  $\lambda^{-1} + \lambda^2 \|\varrho_0\|_{C^1(\mathcal{M})}$  in  $\lambda$ , we get

$$\begin{aligned} \|\varrho\|_{L^2(\mathcal{M})}^2 &\lesssim \|\varrho_0\|_{C^1(\mathcal{M})}^{13/6} + \|\varrho_0\|_{C^1(\mathcal{M})} \|\Lambda_g - \Lambda_{cg}\|^{1/2} \\ &\lesssim \varepsilon^{1/12} \|\varrho_0\|_{C^1(\mathcal{M})}^{25/12} + \varepsilon \|\Lambda_g - \Lambda_{cg}\|^{1/2}. \end{aligned}$$

Since

$$\begin{aligned} \|\varrho_0\|_{C^1(\mathcal{M})} &\lesssim \|\varrho_0\|_{H^{n/2+1+\epsilon}(\mathcal{M})} \\ &\lesssim \|\varrho_0\|_{L^2(\mathcal{M})}^{24/25} \|\varrho_0\|_{H^s(\mathcal{M})}^{1/25} \lesssim \|\varrho_0\|_{L^2(\mathcal{M})}^{24/25} \end{aligned}$$

we therefore obtain

$$\|\varrho\|_{L^2(\mathcal{M})}^2 \lesssim \varepsilon^{1/12} \|\varrho_0\|_{L^2(\mathcal{M})}^2 + \|\Lambda_g - \Lambda_{cg}\|^{1/2}.$$

Taking  $\varepsilon > 0$  small enough we conclude and obtain (1.12).

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